STEMS Physics 2024
Recommended marking scheme

Arjun Maneesh Agarwal

January
2024

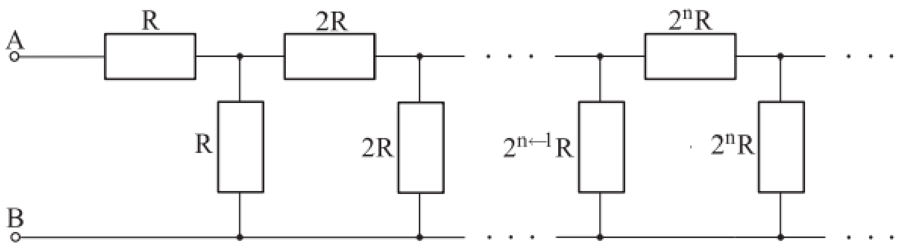
Remark: The general philosophy of this marking scheme follows that of IMO(for the subjective portion). This scheme encourages complete solutions. Partial credits will be given under more strict circumstances. While graders are encouraged to be lenient, do not give sympathy marks or marks for long answers. You may grade a solution in two ways (1) from 7 going down (a complete solution with possible minor errors); (2) from 0 going up (a solution missing at least one critical idea.) We encourage the graders to choose the method favorable to the candidate. If calculation errors or method errors render a part of the solution below 0, award 0 instead.
If you are confused in any part of the key or are grading a solution and the marks are not according to possible point totals, ask someone!
All the participants can challenge the key by emailing their arguments at stemsphysics@gmail.com
You can challenge your scores once they are realized via mail.

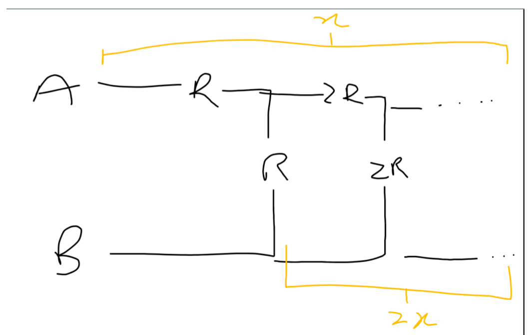
Objective Questions

All objective questions have only possible scores 3 or 0.

1

Find the resistance (in Ω), upto 3 significant figures, between the terminals A and B for the infinite chain shown below. Given





$R = 1\Omega$, the resistances are as shown and increase by a factor of two for each consecutive link.

A twist on a classic...

The only accepted answer is 1.781**.**

This main idea is if for a given circuit all resistance are doubled, the the equivalent resistance is also doubled.
Notice the structure is somewhat recursive, so if we let the answer be x, then we have
Thus,

$$x = 1 + \frac{2x}{2x+1}$$

$$\implies x = \frac{3 \pm \sqrt{17}}{4}$$

As resistance is always positive, $x = \frac{3+\sqrt{17}}{4}$.

2

We have a setup with one giant basketball and two tennis balls with their centers connected with string as shown below. As we let the tennis balls to their sides, at what angle below the center of the basketball will the basketball jump up (if at all)

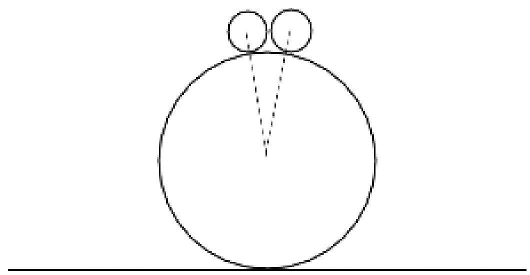


Figure 1: Tennis balls connected to a basketball

Vedant likes confusing mechanics...

The answer is it doesn't jump at all.

The simplest explanation is the principle that frictionless constraint forces (like the ones holding the tennis ball to the basketball) do no work. This is because the angle between the motion and the constraint forces is always 90°, and so the dot product is zero at all times.

3

We have a drop of water of volume $2cm^3$ and has charge Q on it. The drop is cut into half, resulting in two drops of equal radii and charge $Q/2$ each. Considering, negligible losses to surrounding, what should the value of Q (in C), upto 3 significant figures, be so that the drops turn to vapor almost immediately? Take latent

heat for vapourization of water as $2260kj/kg$. We start at lab conditions so the initial temperature is $300K$. Specific heat of water is $4186kj/K * kg$.

Arjun's extremely bad method of heating water.

Answers will be accepted in the range $0.02C$ to $0.06C$.

The energy due to charge on a sphere of radius r with charge density ρ is

$$U = \frac{4\pi\rho^2 r^5}{15\epsilon_0}$$

For derivation check out this Feynman lecture.
As the drops are split into two equal parts $2r'^3 = r^3 \implies r' = \frac{r}{\sqrt[3]{2}}$, and we need to heat water $73K$ and then vaporize it, we have

$$73 * (2260 + 4186) * 2 = \frac{4\pi\rho^2 r^5}{15\epsilon_0} - 2 \cdot \frac{4\pi\rho^2 (\frac{r}{\sqrt[3]{2}})^5}{15\epsilon_0}$$

$$\implies 928226 = \frac{4\pi\rho^2 r^5}{15\epsilon_0}(1 - 2^{\frac{-5}{3}})$$

$$\implies 928226 = \frac{3}{5}\frac{kQ^2}{r}(1 - 2^{\frac{-5}{3}})$$

Notice $\frac{4}{3}\pi\left(\frac{r}{100}\right)^3 = 2 \implies r^3 = 100^3 \frac{3}{2\pi}$.
This all solves to give $Q \approx 0.0377C$.

Note: While this might seem to be very little charge, it is more than enough to cause water to have a dielectric breakdown and not allow further charge to be accommodated. Thus, this is only a theoretical question, and can't be done practically.

4

Making popcorn is quite simple. Just throw dried corn kernels with butter in a utensil and (optionally) shake it till popping subsides. While most people do it by the ear, some chefs like to be

exact. Chef Patrick O Connell's michlin star popcorn is popped till there is a 1.5 sec gap between the pops. Considering, popcorn popping follows a normal distribution, what is time t (in s), upto 3 significant figures, such that once there is a t second gap between pops, 90 percentage of popcorn is popped. Consider an average popcorn takes 30 seconds to pop with a standard deviation of 15 seconds and that an average packet has 100 popcorn.

A few nibbles of Pop science

Answers between $0.75 - 1s$ will be taken as correct.

This question is based on this video by Physics For Birds.
Let the normal distribution of the popcorn be the function f and thus, $f(\tau)$ be the probability that a popcorn pops exactly at time τ. For some small interval of time dt starting at some time τ, (expected) number of pops in that interval, given we started with N_0 popcorn, is

$$N_\tau = N_0 f(\tau) dt$$

We are able to make the assumption $\int_\tau^{\tau+dt} f(x)dx = f(\tau)dt$ as dt is small.
Time between pops at τ is clearly

$$g(\tau) = \frac{dt}{N_\tau} = \frac{1}{N_0 f(\tau)}$$

We want to stop at τ such that 90% popcorn is popped, that is

$$\int_0^\tau f(t)dt \geq 0.9$$

Putting in

$$f(t) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-30}{15}\right)}$$

where t is in seconds, we will get $\tau \approx 49.234375\ldots$.
This implies $g(\tau) = 0.85$.

Note: Again, a small thing which can be left for further analysis is that there is a subsequent normal distribution going one regarding the burning of popcorn. Also, not all popcorn pops. See the video mentioned for a more through analysis which also agrees with Chef O Conell's number.

Note, the original question had ambiguous wording. Upto the interpretation of the reader, this confused the answer between τ and $g(\tau)$. Thus, the an alternate correct answer was the values from $45 - 55$.

5

Physicists are smart but superstitious folks. You might have noticed that when you remove your coat during the winter, static electricity forms, and you feel small shocks. Some physicists humorously believe that thunder occurs when Zeus takes his coat off. Now, consider that when you get a jacket shock, it's at $750V$, while thunder happens at 10 million volts. Also assume that your skin and Zeus's skin have the same work function and that your coat and Zeus's coat is made of the same material. Assuming Zeus is a human scaled by a size factor s and the force he applies to take the coat off scales accordingly, what is the value of s, upto 3 significant figures?

Arjun got a coat shock.

Answers are accepted from range $20 - 25$.

One can clearly see that the potential difference generated by the removal of coat should depend on the work function of the skin and coat, the force used to remove the coat, the velocity of the coat coming off and the area of contact.

A very important observation here is that the capacitance between the jacket and the skin plays a role. A rule of thumb is *'heat is due to resistors, shocks due to capacitors and magnetization due to inductance.'*. Notice the dimensions, We can represent this as

$$V = w^{\alpha} F^{\beta} v^{\gamma} C^{\delta}$$

V	Shock Voltage	$ML^2T^{-3}I^{-1}$
w	Work function of skin or coat	ML^2T^{-2}
F	Force to remove coat	MLT^{-2}
v	Velocity to remove coat	LT^{-1}
C	Capacitance	$M^{-1}L^{-2}T^4I^2$

Making the equations, one gets

$$M : \alpha + \beta - \delta = 1$$
$$L : 2\alpha + \beta + \gamma - 2\delta = 2$$
$$T : -2\alpha - 2\beta - \gamma + 4\delta = -3$$
$$I : 2\delta = -1$$

This tells us

$$\delta = \frac{-1}{2}$$
$$\gamma = 2$$
$$\alpha = \frac{-3}{2}$$
$$\beta = 2$$

Thus,

$$V = kw^{\frac{-3}{2}}F^2v^2C^{\frac{-1}{2}}$$

We already know that the work function is not changing. The energy delivered by muscles scales by s^3 as it depends on muscle mass. This implies velocity scales by s^0 as mass scaled by s^3.
As velocity is not changing, time for the coat to come out is scales by s as it depends on the length of back. This implies that the power delivered by the muscles scales by s^2. As velocity scaled by s^0, force scales by s^2.
Capacitence depends on space between coat and skin, permitivity and area. As the space and permitivity have no reason to change, capacitence scales by s^2.
Thus, V scales by s^3.
Thus, $\frac{10^7}{750} = s^3 \implies s = 23.713$.

6

A small ball is placed along the axis of a concave mirror of focal length $f = 10cm$ at an object distance $u = 12cm$. The concave surface of the mirror is filled with a thin layer of liquid of refractive index $n = 1.5$. If the image of the ball is formed by the rays impinging on the mirror near its vertex, determine the location of the image(distance from vertex, following sign convention), in cm, upto 3 significant figures.

An olympiad trick appears in the wild.

The only accepted answer is -15 cm.

This question is borrowed from Jinhui Wang's Competitive Physics, volume 2.
The main idea is that this setup has it's own new focus. This can be observed by imagining a bunch of parallel lines on the mirror. They initially pass by the liquid film undeviated but undergo refraction at the surface of the film after being reflected by the mirror — causing them to converge at a distance $\frac{f}{n}$ where f is the focal distance and n is the refractive index.
Applying the mirror equation will give us the final distance as

$$v = \frac{fu}{nu - f}$$

which in this question gives us $v = -15cm$.

7

The Bekenstein–Hawking formula expresses the entropy of a black hole as:

$$S = \frac{A}{4}$$

where A represents the area of the event horizon. For an uncharged, non-rotating black hole, the radius is given by:

$$R = 2M$$

Here, the constants $\hbar$, c, and G are all set to unity, so no need to restore these factors. Consider two uncharged, non-rotating black holes that start very far apart and eventually merge into a single black hole, emitting gravitational waves during the process. Assume no initial angular momentum, meaning that the resulting black hole is also non-rotating. Your task is to determine the maximum possible efficiency of this process(upto 3 significant figures), which is defined as the fraction of the initial energy converted into gravitational waves, for any set of initial black hole masses.

Black hole merger!

The only accepted answer is 0.293

This question was borrowed from a Kevin Zhou problem set. The solution (and further discusion) can be found here.

1 Subjective

1.1 1

A relativistic particle of charge q is accelerated with a uniform electric field E in a region where a perpendicular uniform magnetic field B is also present. Derive the trajectory of the particle if the initial velocity v starts at an angle θ with E. Do not try to get a coordinate form. It is nasty. A differential form will be given full credit.

Charges are strange when they go fast!

The differential equation is given by Lorentz Law of electromagnetism. Surprisingly, the equation doesn't change when we switch from non-relativistic frame to a relativistic system.

$$m\frac{d^2}{dt^2}\vec{X} = q(\vec{E} + \frac{d}{dt}\vec{X} \times \vec{B})$$
$$\Rightarrow m\frac{d^2}{dt^2}\vec{X} = q(E\sin\theta\hat{x} + E\cos\theta\hat{y} + \frac{d}{dt}\vec{X} \times (B\hat{z}))$$

Even when the final equation is not given, 2 credits are given for observing that the particle stays in the "x-y plane".
A more in-depth explanation of this without the θ dependence is given here

1.2 2

When you pour a fizzy drink into a glass and gently tap the base, you notice an intriguing phenomenon: with every successive tap, even if you pause for a moment, the pitch steadily increases. This happens because escaping carbonation alters the compressibility of the drink. Assuming the soft drink shares the same density and compressibility as water and that the wavelength of the sound produced is much larger than the radius or spacing of the bubbles, derive a formula to determine the level of carbonation in the drink (volume of CO_2 per liter of drink) based on the tap frequency f_0 observed when the drink has gone flat.

Hot Chocolate effect on cold drinks

We will use the relation

$$v = \sqrt{\frac{\beta}{\rho}}$$

where β is the bulk modulus and ρ is the mass density.
We already know

$$\beta = V\frac{dP}{dV}$$

Traditionally, there is a $-$ sign here. We will ignore it for convenience.

Mentioning both the formulas in any form will be worth 0.5 **marks each,**

As both water and air are somewhat compressible. So $\frac{dP}{dV}$ term is negative. Notice $V = V_w + V_c$ where V_w is the volume of water and V_c is the volume of CO_2. This implies, $dV = dV_w + dV_c$. Also as the density of water is much higher than density of CO_2, $\rho = \rho_w$. The pressure is same on the cold-drink, so same on both the components.

$$v^2 = \frac{V_w + V_c}{\rho_w}\frac{dP}{dV_w + dV_c)}$$

$$\implies \frac{1}{v^2} = \frac{\rho_w}{V_w + V_c}\left(\frac{dV_w}{dP} + \frac{dV_c}{dP}\right)$$

Let's take $F = \frac{V_c}{V_w}$ which implies $V_c = fV_w$.

$$\implies \frac{1}{v^2} = \frac{\rho_w}{(1+F)V_w}\left(\frac{dV_w}{dP} + \frac{dV_c}{dP}\right)$$

Notice, $v_w = \frac{v_w}{\rho_w}\frac{dP}{dV_w}$ and $\frac{1}{\beta_c} = \frac{1}{V_c}\frac{dV_c}{dP_c}$

$$\implies \frac{1}{v^2} = \frac{1}{(1+F)v_w^2} + \frac{\rho_w}{\beta_c}\frac{F}{1+F}$$

$$\implies \frac{1}{v^2} + \frac{F}{v^2} = \frac{1}{v_w^2} + \frac{\rho_w}{\beta_c}F$$

$$\implies \left(\frac{1}{v^2} - \frac{1}{v_w^2}\right) = F\left(\frac{\rho_w}{\beta_c} - \frac{1}{v^2}\right)$$

$$F = \frac{(v_w^2 - v^2)\beta_c}{v_w^2(v^2\rho_w - \beta_c)}$$

Getting the expression of F is worth 5 marks. Be lenient and don't deduct marks here for small errors.

The fundamental frequency of the tap is $f = \frac{v}{4h}$ where h is the height till which we have filled the soft drink. Note, bottom of the glass is 'open' with respect to this case as our medium is the liquid.
Also note, the height in the glass will not change significantly, The CO_2 is dissolved in water in two forms. One by getting 'embedded' spaces between the water molecules. Check out partial molar volumes.
The other is due to the reaction to form carbonic acid. This is what makes seltzer water taste a little different than regular water. As water and carbonic acid hardly have much of a molecular volume difference due to very similar bonding structure, the height doesn't change much. Thus, as h is not changing throughout the experiment, $v_w = 4hf_0$ as that is the case when it is flat. Similerly $v = 4hf$.

$$\implies F = \frac{(f_0^2 - f^2)\beta_c}{f_0^2(16h^2f^2\rho_w - \beta_c)}$$

Now notice, we want the amount of CO_2 per liter of soft drink and we already showed that the volume of soft drink hardly changed during the process, thus F is our answer.

Giving any justifiable reason for the height not changing and reaching the final form is worth 1 mark. Award 1 mark if the explanation is correct but the algebra is wrong. Award 0.5 marks if the algebra is correct but the explanation is wrong or missing.

This question is based on a slightly simpler result called the 'The hot chocolate effect' by Frank S. Crawford.

1.3 3

Calculate the pressure exerted on the walls of a small cubic box with edge length d by a neutron contained within the box.

Harshita's favorite from Gnadig...

Solution 1

This particle can not be at rest due to quantum effects and must have a linear momentum of *average* magnitude p. According to Heisenberg's uncertainty principle,

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

where $\Delta x \approx d$ is the uncertainty in the neutron's position and $\Delta p \approx p$ is the uncertainty in linear momentum. The average value of the (vector) linear momentum must be zero because the neutron cannot leave the box. This is why the linear momentum's uncertainty (the mean deviation from the average value) can be estimated as being equal to its (scalar) magnitude. For the sake of definiteness, consider the limiting case of the uncertainty relationship in which equality holds. The magnitude of linear momentum of the neutron is

$$p \approx \pm \frac{\hbar}{2d}$$

and so, its speed will be

$$v \approx \pm \frac{\hbar}{2md}.$$

This can be interpreted as the neutron bouncing back and forth between the opposite walls of the box. So, at one of the walls, a change of linear momentum of $2mv$ occurs at regular intervals of $\Delta t = 2d/v$, and as a result, the particle exerts a force of

$$F = \frac{2mv}{\Delta t} = \frac{mv^2}{d} = \frac{\hbar^2}{4md^3}$$

on the wall. The corresponding pressure is

$$P = \frac{F}{d^2} = \frac{\hbar^2}{4md^5}$$

Solution 2
The ground state of a particle enclosed in a box with edges of length d is described by a wave whose half-wavelength is equal to d $(\lambda/2 = d)$. The de Broglie relations give the particle's linear momentum as

$$p_x = p_y = p_z = \frac{h}{\lambda} = \frac{h}{2d}.$$

The kinetic energy of such a particle is

$$E = \frac{1}{2}(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

This becomes

$$E(d) = \frac{3h^2}{8md^2}$$

Now suppose the edges of the box are slowly reduced by an amount $\Delta d \ll d$; its volume consequently decreases by

$$\Delta V = d^3 - (d - \Delta d)^3 \approx 3d^2 \Delta d.$$

The work required to bring this about is

$$W = P\Delta V,$$

Where P is the quantum pressure to be determined. The corresponding increase in the particle's energy is

$$\Delta E = E(d - \Delta d) - E(d).$$

From the work-energy theorem ($W = \Delta E$),

$$P \cdot 3d^2 \Delta d = \frac{3h^2}{8m} \Delta \left(\frac{-1}{d^2} \right) \approx \frac{3h^2}{4md^3} \Delta d$$

We see that the required pressure is

$$P = \frac{h^2}{4md^5}$$

Rubric for question 3

Solution 1 (Uncertainty method)
1. Initial setup using uncertainty principle [2 points]

- Writing $\Delta x \Delta p \geq \frac{\hbar}{2}$ [0.5]
- Justifying $\Delta x \approx d$ and $\Delta p \approx p$ [1.0]
- Explaining why average momentum is zero [0.5]

2. Finding momentum and velocity [2 points]

- Deriving $p \approx \pm \frac{\hbar}{2d}$ [1.0]
- Converting to velocity $v \approx \pm \frac{\hbar}{2md}$ [1.0]

3. Force and pressure calculation [3 points]

- Setting up force using $F = \frac{2mv}{\Delta t}$ [1.0]
- Deriving $F = \frac{\hbar^2}{4md^3}$ [1.0]
- Final pressure $P = \frac{\hbar^2}{4md^5}$ [1.0]

Solution 2 (Energy Method)

1. Wave function analysis [2 points]

- Using $\lambda/2 = d$ condition [0.5]
- Deriving momentum using de Broglie relation [1.5]

2. Energy calculation [2 points]

- Writing kinetic energy equation [0.5]
- Deriving $E(d) = \frac{3h^2}{8md^2}$ [1.5]

3. Pressure derivation [3 points]

- Volume change analysis [1.0]
- Work-energy theorem application [1.0]
- Final pressure derivation [1.0]

General Notes:

- Both solutions are equally valid and should receive the same maximum score
- Missing physical justifications: -1 point per instance
- Sign errors: -0.5 points if method is correct
- Dimensional errors: no credit for that step
- Alternate valid approaches will be considered and marked accordingly
- No credit for stating formulas without proper derivation or physical reasoning

Calculate the pressure exerted on the walls of a small cubic box with edge length d by a neutron contained within the box.

Solution 1 This particle can not be at rest due to quantum effects and must have a linear momentum of *average* magnitude p. According to Heisenberg's uncertainty principle,

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

where $\Delta x \approx d$ is the uncertainty in the neutron's position and $\Delta p \approx p$ is the uncertainty in linear momentum. The average value of the (vector) linear momentum must be zero because the neutron cannot leave the box. This is why the linear momentum's uncertainty (the mean deviation from the average value) can be estimated as being equal to its (scalar) magnitude. For the sake of definiteness, consider the limiting case of the uncertainty relationship in which equality holds. The magnitude of linear momentum of the neutron is

$$p \approx \pm\frac{\hbar}{2d}$$

and so, its speed will be

$$v \approx \pm\frac{\hbar}{2md}.$$

This can be interpreted as the neutron bouncing back and forth between the opposite walls of the box. So, at one of the walls, a change of linear momentum of $2mv$ occurs at regular intervals of $\Delta t = 2d/v$, and as a result, the particle exerts a force of

$$F = \frac{2mv}{\Delta t} = \frac{mv^2}{d} = \frac{\hbar^2}{4md^3}$$

on the wall. The corresponding pressure is

$$P = \frac{F}{d^2} = \frac{\hbar^2}{4md^5}$$

Solution 2 The ground state of a particle enclosed in a box with edges of length d is described by a wave whose half-wavelength is equal to d ($\lambda/2 = d$). The de Broglie relations give the particle's linear momentum as

$$p_x = p_y = p_z = \frac{h}{\lambda} = \frac{h}{2d}.$$

The kinetic energy of such a particle is

$$E = \frac{1}{2}(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

This becomes

$$E(d) = \frac{3h^2}{8md^2}$$

Now suppose the edges of the box are slowly reduced by an amount $\Delta d \ll d$; its volume consequently decreases by

$$\Delta V = d^3 - (d - \Delta d)^3 \approx 3d^2\Delta d.$$

The work required to bring this about is

$$W = P\Delta V,$$

Where P is the quantum pressure to be determined. The corresponding increase in the particle's energy is

$$\Delta E = E(d - \Delta d) - E(d).$$

From the work-energy theorem ($W = \Delta E$),

$$P \cdot 3d^2 \Delta d = \frac{3h^2}{8m} \Delta \left(\frac{-1}{d^2} \right) \approx \frac{3h^2}{4md^3} \Delta d$$

We see that the required pressure is

$$P = \frac{h^2}{4md^5}$$