TESSELLATE PRESENTS

STEMS

Scholastic Test of Excellence in Mathematical Sciences

Subject Category C

Exam Date : 17th December, 2023
Exam Timing : 11:00 AM IST - 5:00 PM IST

Rules and Regulations

Marking Scheme

1. The question paper contains **thirteen** questions, **seven** integer type questions (**Part A**), and **six** subjective questions (**Part B**).
2. Each subjective question is worth **7 marks**. Each objective question is worth **3 marks**.
3. A candidate's submission for the **Part B** of the exam will be checked only if they are in the top 15 candidates for **Part A**.
4. Time duration is **6 hours: 11:00 AM IST - 5:00 PM IST**.
   Submit your answers on the google form given below by **5:30 PM IST**.

Miscellaneous

1. Use the google form: **https://forms.gle/dXzihFvtPdADQMbW6** , to submit your answers.
2. For **Part A**, give answers in the form of a single integer, without any whitespaces, commas, periods, semi-colons, underscores or any other special characters. Submissions with special characters such as these will NOT be graded (hyphens are allowed for negative integers).
3. For **Part B**, you can either LaTeX or handwrite your solutions neatly.
   Submit a PDF file (either scanned or LaTeXed) **ONLY**. No other form of file submission will be accepted. Name your file **"math_rollnumber"** (here rollnumber is the 5 digit schoolpay/airpay receipt number generated at the time of registration). Use the last 5 digits of your roll number. If your roll number is stems2023XXXXX, name your file math_XXXXX
4. Make sure to keep the file size below the 10 MB limit. You can use online file compression services in case your file size exceeds 10 MB.
5. Use a good application to scan handwritten text into PDF. Kindly make sure that the answers are legible and that your furniture or flooring is not a part of the submission.
6. Solutions should be brief and should contain all the necessary details. Ambiguous or illegible answers will not gain credits. If you strike something out, strike it out properly so that it is clear to the evaluator what you want to be read. Please avoid overwriting your answers.
7. Do **NOT** post/share the questions appearing in the contest on any forums or discussion groups while the contest is live. It will result in immediate disqualification of involved candidates when caught.
8. Answers should be your own and should reflect your independent thinking process. Any form of plagiarism or failure to comply with aforementioned regulations may lead to disqualification.

Contact details

- For subject related queries, clearly mention your **category (A/B/C)** in the mail or WhatsApp text.
- For **subject related** queries, contact:
  - Official email ID: **stemsmath2024@gmail.com**

- Chandralekha
  Email : **chandralekha@cmi.ac.in**
  WhatsApp: +91 73387 18671
- Krutarth Shah
  Email: **krutarth.ug2023@cmi.ac.in**
- Sunaina Pati
  Email : **sunaina.ug2023@cmi.ac.in**
  WhatsApp: +91 70990 99967

Note: Use the personal emails only if the official email is unreachable. Use WhatsApp only if absolutely necessary, otherwise email is preferred.

- For **technical** queries, contact:
  - Official email ID: **tessellate.cmi@gmail.com**
  - Harini Rammohan
    Email: **harini@cmi.ac.in**
  - Ananya Ranade
    Email: **ananyar@cmi.ac.in**

Questions

Part A

1. **[3 marks]**

   Let
   $$N = \sum_{i=0}^{2023^9-1} 7i(i+1)(i^2+i+1)^2.$$
   Let $a = \nu_3(N)$. Find a^5.
   Notation: Let p be a prime and n be a positive number. Then $\nu_p(n)$ is the exponent of p in the prime factorization of n. That is, if $\nu_p(n) = k$ then $p^k | n, p^{k+1} \nmid n$

2. **[3 marks]** The numbers $1, 2, \ldots, 100$ are written on a board. Every minute, Sheldon selects four numbers a, b, c, d on the board, erases them, and then writes $(a^3 + b^3 + c^3 + d^3)^{1/3}$ in their place. He continues to do that until no more numbers can be erased. Let the largest possible number that can remain on the board be N. What is N^3?

3. **[3 marks]** Let ABC be a triangle with I_A, I_B, I_C as the A, B, C excenters respectively and I the incentre. Let O be the circumcenter of ABC. Let circumcircle (ICI_B) and circumcircle $(I_AI_BI_C)$ intersect at $E \neq I_B$. Let $EI \cap (ABC) = F$ with F and A lying on opposite sides of BC. Suppose $IF = 17, IO = 23, OC = 40$ and $IE = \frac{a+b\sqrt{c}}{d}$, where a, b, c, d are nonnegative integers such that c is square-free and $\gcd(a, d) = 1$. Find a.

4. **[3 marks]** Numbers $1, 2, \ldots, 10$ are written on a blackboard. Alice makes 5 tuples from these numbers such that 2 numbers are in each tuple, and each number is in exactly 1 tuple. She then picks the minimum of the 2 numbers in each tuple, and adds up the 5 numbers obtained. Let S be the expected value of this sum. Let $S = \frac{a}{b}$ where $(a, b) = 1$. Find $a + b$

5. **[3 marks]** Given $f : \mathbb{N} \mapsto \mathbb{N}$ is a function satisfying $f(1) = 1$ and that for all $n > 1$ there are distinct divisors d_1, d_2 of n satisfying
   $$n = f(d_1) + f(d_2)$$
   Find the sum of all possible values of $f(2023)$.

6. **[3 mark]** Let $P(x) = 2x^3 - 6069x^2 + 4092529x + \frac{2023}{2}$. Let
   $$\frac{a}{b} = \int_0^{2023} P^{2023}(x)\ \mathrm{d}x.$$
   where $(a, b) = 1$. Find $10a + b$.
   Notation: $P^n(x) = P(P^{n-1}(x))$ where $P^2(x) = P(P(x))$.

7. **[3 marks]** Given a quadrilateral $ABCD$ such that $\angle ACD = 66°$, $\angle ADB = 51°$, $\angle BDC = 21°$ and $\angle ACB = 48°$. Suppose $AD = a, BC = b, AC = d$ then find the value of $100\frac{BD}{\sqrt{b(a+d+b)}} + 2$.

Part B

1. **[7 marks]** Evaluate

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{2^i}{i}}{\frac{2^n}{n}}$$

2. **[7 marks]** Does there exist a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that the image of every uncountable subset of $\mathbb{R}$ is unbounded?

3. **[7 marks]** All the rationals are coloured with n colours so that, if rationals a and b are colored with different colours then $\frac{a+b}{2}$ is coloured with a colour different from both a and b. Prove that every rational is coloured with the same colour.

4. **[7 marks]** Let ABC be an acute triangle. Let P be a point in the plane.
In a move, you can reflect the point across any side of the triangle. Is it true that for all points P, there is a finite sequence of moves after which the point is inside the triangle?

5. **[7 marks]** Find the sum of all primes $p < 50$, for which there exists a function $f : \{0, \dots, p-1\} \mapsto \{0, \dots, p-1\}$ such that $p \mid f(f(x)) - x^2$

6. **[7 marks]** Let $A, B \in M_n(\mathbb{Q})$ be $n \times n$ matrices such that the characteristic polynomials of A and B are equal and irreducible. Let $V = \{M | M \in M_n(\mathbb{Q}), AM = MB\}$ be a vector space over $\mathbb{Q}$. Prove that $\dim V = n$.

END OF QUESTION PAPER