TESSELLATE PRESENTS

STEMS

Scholastic Test of Excellence in Mathematical Sciences

Subject Category A

Exam Date	**:**	**17th December, 2023**
Exam Timing	**:**	**11:00 AM IST - 5:00 PM IST**

Rules and Regulations

Marking Scheme

1. The question paper contains **thirteen** questions, **seven** integer type questions (**Part A**), and **six** subjective questions (**Part B**).
2. Each subjective question is worth **7 marks**, and each integer type question is worth **3 marks**.
3. A candidate's submission for the **Part B** of the exam will be checked only if they are in the top 30 candidates for **Part A**.
4. Time duration is **6 hours: 11:00 AM IST - 5:00 PM IST**.
   Submit your answers on the google form given below by **5:30 PM IST**.

Miscellaneous

1. Use the google form: **https://forms.gle/9knhZfNswrP8F2YS8** , to submit your answers.
2. For **Part A**, give answers in the form of a single integer, without any whitespaces, commas, periods, semi-colons, underscores or any other special characters. Submissions with special characters such as these will NOT be graded (hyphens are allowed for negative integers).
3. For **Part B**, you can either LaTeX or handwrite your solutions neatly.
   Submit a PDF file (either scanned or LaTeXed) **ONLY**. No other form of file submission will be accepted. Name your file **"math_rollnumber"** (here rollnumber is the 5 digit schoolpay/airpay receipt number generated at the time of registration). Use the last 5 digits of your roll number. If your roll number is stems2023XXXXX, name your file math_XXXXX
4. Make sure to keep the file size below the 10 MB limit. You can use online file compression services in case your file size exceeds 10 MB.
5. Use a good application to scan handwritten text into PDF. Kindly make sure that the answers are legible and that your furniture or flooring is not a part of the submission.
6. Solutions should be brief and should contain all the necessary details. Ambiguous or illegible answers will not gain credits. If you strike something out, strike it out properly so that it is clear to the evaluator what you want to be read. Please avoid overwriting your answers.
7. Do **NOT** post/share the questions appearing in the contest on any forums or discussion groups while the contest is live. It will result in immediate disqualification of involved candidates when caught.
8. Answers should be your own and should reflect your independent thinking process. Any form of plagiarism or failure to comply with aforementioned regulations may lead to disqualification.

Contact details

- For subject related queries, clearly mention your **category (A/B/C)** in the mail or WhatsApp text.
- For **subject related** queries, contact:
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Note: Use the personal emails only if the official email is unreachable. Use WhatsApp only if absolutely necessary, otherwise email is preferred.

- For **technical** queries, contact:
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Questions

Part A

1. **[3 marks]** Let

$$N = \sum_{i=0}^{2023^9-1} 7i(i+1)(i^2+i+1)^2.$$

Let $a = \nu_3(N)$. Find a^5.
Notation: Let p be a prime and n be a positive number. Then $\nu_p(n)$ is the exponent of p in the prime factorization of n. That is, if $\nu_p(n) = k$ then $p^k | n, p^{k+1} \nmid n$

2. **[3 marks]** The numbers $1, 2, \ldots, 100$ are written on a board. Every minute, Sheldon selects four numbers a, b, c, d on the board, erases them, and then writes $(a^3 + b^3 + c^3 + d^3)^{1/3}$ in their place. He continues to do that until no more numbers can be erased. Let the largest possible number that can remain on the board be N. What is N^3?

3. **[3 marks]**Let ABC be a triangle with I_A, I_B, I_C as the A, B, C excenters respectively and I the incentre. Let O be the circumcenter of ABC. Let circumcircle (ICI_B) and circumcircle $(I_AI_BI_C)$ intersect at $E \neq I_B$. Let $EI \cap (ABC) = F$ with F and A lying on opposite sides of BC. Suppose $IF = 17, IO = 23, OC = 40$ and $IE = \frac{a+b\sqrt{c}}{d}$, where a, b, c, d are nonnegative integers such that c is square-free and $\gcd(a, d) = 1$. Find a.

4. **[3 marks]** The roots in $\mathbb{C}$ of $x^9 - 2x^8 + 3x^7 + 5x^5 - 2x^2 + 3x - 6$, are permuted and written on a board. The following operation is repeated till there is a single number on the board s: two adjacent numbers x and y are chosen and replaced with $\frac{xy}{x+y}$. Find the sum of all distinct s for all such permutations of the roots.

5. **[3 marks]** Numbers $1, 2, \ldots, 10$ are written on a blackboard. Alice makes 5 tuples from these numbers such that 2 numbers are in each tuple, and each number is in exactly 1 tuple. She then picks the minimum of the 2 numbers in each tuple, and adds up the 5 numbers obtained. Let S be the expected value of this sum. Let $S = \frac{a}{b}$ where $\gcd(a, b) = 1$. Find $a + b$

6. **[3 marks]** Given $f : \mathbb{N} \mapsto \mathbb{N}$ is a function satisfying $f(1) = 1$ and that for all $n > 1$ there are distinct divisors d_1, d_2 of n satisfying

$$n = f(d_1) + f(d_2)$$

Find the sum of all possible values of $f(2023)$.

7. **[3 mark]** Given a quadrilateral $ABCD$ such that $\angle ACD = 66°$, $\angle ADB = 51°$, $\angle BDC = 21°$ and $\angle ACB = 48°$. Suppose $AD = a, BC = b, AC = d$ then find the value of $100\frac{BD}{\sqrt{b(a+d+b)}} + 2$.

Part B

1. **[7 marks]** Let n be a positive integer and $S = \{m \mid 2^n \leq m < 2^{n+1}\}$. We call a pair of non-negative integers (a, b) *fancy* if $a + b$ is in S and is a palindrome in binary. Find the number of fancy pairs (a, b).

2. **[7 marks]** Let $S = \mathbb{Z} \times \mathbb{Z}$. A subset P of S is called *nice* if

   - $(a, b) \in P \implies (b, a) \in P$
   - $(a, b), (c, d) \in P \implies (a + c, b - d) \in P$

   Find all $(p, q) \in S$ so that if $(p, q) \in P$ for some nice set P then $P = S$.

3. **[7 marks]** Let ABC be a triangle. Let I be the Incenter of ABC and S be the midpoint of arc BAC. Define I_A as the A−excenter wrt ABC. Define ω to be the circle centred at S with radius SB. Let $AI_A \cap \omega = X, Y$. Show that $\angle BCX = \angle ACY$.

4. **[7 marks]** In CMI, each person has atmost 3 friends. A disease has infected exactly 2023 people in CMI . Each day, a person gets infected if and only if atleast two of their friends were infected on the previous day. Once someone is infected, they can neither die nor be cured. Given that everyone in CMI eventually got infected, what is the maximum possible number of people in CMI?

5. **[7 marks]** Let r, s be real numbers, find maximum t so that if $a_1, a_2, \ldots$ is a sequence of positive real numbers satisfying

   $$a_1^r + a_2^r + \cdots + a_n^r \leq 2023 \cdot n^t$$

   for all $n \geq 2023$ then the sum

   $$b_n = \frac{1}{a_1^s} + \cdots + \frac{1}{a_n^s}$$

   is unbounded, i.e for all positive reals M there is an n such that $b_n > M$.

6. **[7 marks]** Let ABC with orthocenter H and circumcenter O be an acute scalene triangle satisfying $AB = AM$ where M is the midpoint of BC. Suppose Q and K are points on (ABC) distinct from A satisfying $\angle AQH = 90°$ and $\angle BAK = \angle CAM$. Let N be the midpoint of AH.

   - Let I be the intersection of B−midline and A−altitude Prove that $IN = IO$.
   - Prove that there is point P on the symmedian lying on circle with center B and radius BM such that (APN) is tangent to AB.

END OF QUESTION PAPER