TESSELLATE PRESENTS

STEMS

Scholastic Test of Excellence in Mathematical Sciences

Computer Science Category B

Exam Date : 16th December, 2023
Exam Timing : 3:00 PM IST - 6:00 PM IST

Rules and Regulations

Marking Scheme

1. The question paper contains **twelve** questions, **seven** numeric type questions (**Part A**), and **five** subjective questions (**Part B**).
2. Each subjective question is worth **7 marks**. Each numeric type question is worth **3 marks**.
3. A candidate's submission for the **Part B** of the exam will be checked only if the total score obtained by the candidate in the **Part A** is **atleast 9 OR** they are in the top 20 candidates for **Part A**.
4. Time duration is **3 hours: 3:00 PM IST - 6:00 PM IST**.
   Submit your answers on the google form given below by **6:30 PM IST**.

Miscellaneous

1. Use the google form: **https://forms.gle/9NAPcwWWhZAnqC9R7** , to submit your answers.
2. For **Part A**, give answers in numeric form, without any whitespaces, commas, periods, semi-colons, underscores or any other special characters. Submissions with special characters such as these will NOT be graded (hyphens are allowed for negative integers).
3. For **Part B**, you can either LaTeX or handwrite your solutions neatly.
   Submit a PDF file (either scanned or LaTeXed) **ONLY**. No other form of file submission will be accepted. Name your file **"cs_rollnumber"** (here rollnumber is the 4 digit schoolpay/airpay receipt number generated at the time of registration).
4. Make sure to keep the file size below the 10 MB limit. You can use online file compression services in case your file size exceeds 10 MB.
5. Use a good application to scan handwritten text into PDF. Kindly make sure that the answers are legible and that your furniture or flooring is not a part of the submission.
6. Solutions should be brief and should contain all the necessary details. Ambiguous or illegible answers will not gain credits. If you strike something out, strike it out properly so that it is clear to the evaluator what you want to be read. Please avoid overwriting your answers.
7. Do **NOT** post/share the questions appearing in the contest on any forums or discussion groups while the contest is live. It will result in immediate disqualification of involved candidates when caught.
8. Answers should be your own and should reflect your independent thinking process. Any form of plagiarism or failure to comply with aforementioned regulations may lead to disqualification.

Contact details

- For subject related queries, clearly mention your **category (A/B)** in the mail or WhatsApp text.
- For **subject related** queries, contact (Kindly CC all the four mails : **stemscs2024@gmail.com** , **amishra@cmi.ac.in**, **vardhan@cmi.ac.in**, **ananyar@cmi.ac.in**):
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Note: Use WhatsApp only if absolutely necessary, otherwise email is preferred. While sending queries through Whatsapp, kindly send messages to all the numbers for prompt reply.

- For **technical** queries, contact:
  - Official email ID: **tessellate.cmi@gmail.com**
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    Email: **siddhants@cmi.ac.in**
  - Ananya Ranade
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Questions

Part A

1. **[3 marks]** Alice starts with the number $n = 2023$ on a blackboard. She can perform the following move: if there's a number k on the board, she can also write either $k - 5$ or $\lfloor k/2 \rfloor$. How many distinct positive numbers can she write on the board through these moves?

2. **[3 marks]** Neena starts at $(1,1)$ on the integer lattice and takes the following random walk: If she is at point (a,b), with probability $1/2$ each, she either:

   - Goes horizontally to $(a+b,b)$
   - Goes vertically to (a,a)

   Given that the expected value of the sum of coordinates after Neena takes her 2023^{rd} vertical step is m, what is the highest power of 2 dividing m?

3. **[3 marks]** Let G be a perfect binary tree (follow the hyperlink for definition) with $2^n - 1$ vertices. A pair of vertices in the final (n^{th} layer) are picked uniformly at random and joined by an edge. Let l be the expected length of the shortest cycle formed when $n = 16$. What is the value of $\lfloor l \rfloor$?

4. **[3 marks]** Suppose X be a random variable which takes the value $(1,0)$ with probability $\frac{2}{3}$ and $(0,1)$ with probability $\frac{1}{3}$. Consider independent and identically distributed random variables $X_1, \ldots, X_{2023} \sim X$. Define $Y = X_1 + \ldots + X_{2023}$. If the expected value of $d(Y, (\frac{2n}{3}, \frac{n}{3}))^2$ is $\frac{a}{b}$, where a and b are positive co-prime integers, what is the value of ab when $n = 2024$?

5. **[3 marks]** Bob has initially written the number 0 on a blackboard. In each move, if the number on the board is n, he can either write $3n, n+1$ or $n-1$. Let $n(x)$ represent the minimum number of moves required for Bob to write the natural number x on the board. Suppose $\mathcal{S}$ be the set of naturals $\leq 3^9$ that Bob can write within 17 moves i.e. $\mathcal{S} = \{x \mid n(x) \leq 17, 1 \leq x \leq 3^9\}$. Determine the cardinality of the set $\mathcal{S}$.

6. **[3 marks]** Order the following 5 functions based on their complexity. The functions are defined below for $n \geq 2023$ and are all 1 for all $n < 2023$.

   - $f_1(n) = 4f_1(n/4) + n$
   - $f_2(n) = 4f_2(n/2) + n^2$
   - $f_3(n) = \sqrt{n} f_3(\sqrt{n}) + n$
   - $f_4(n) = 2f_4(n/2) + 2023$
   - $f_5(n) = f_5(n-1) + f_5(\sqrt{n})$

Write your answer as a single 5 digit integer. (i.e. response 51243 would indicate $f_5(n) > f_1(n) > f_2(n) > f_4(n) > f_3(n)$ for all large enough n.)

7. **[3 marks]** Let the maximum possible value of $w + x + y + z$ in the region $\mathcal{P}$ be $\frac{a+b\sqrt{c}}{d}$ in reduced form where c is square free and a, b are integers and d is a positive integer:

$$\mathcal{P} = \{(w, x, y, z) \mid 0 \leq w, x, y, z \leq 2, w^2 + 2x^2 + 4y^2 + 8z^2 \leq 11\}$$

Find $1000a + 100b + 10c + d$.

Part B

1. [**7 marks**] A set $S \subset \mathbb{F}_2^{2n}$ is called *uwu* if $\forall a \neq b \in S$, $a + b$ is not palindromic. Some $s \in \mathbb{F}_2^{2n}$ is called palindromic if when written out as a string s, is a palindrome.

   For instance, if $n = 2$, take $S = \{(1,0,1,1),(1,1,1,1)\}$; then S is also an *uwu* set as $(1,0,1,1) + (1,1,1,1) = (0,1,0,0)$ which is not a palindrome when written as 0100.

   (a) Find the maximum size of an *uwu* set.

   (b) Prove that if S_1 and S_2 are *uwu* with $|S_1| < |S_2|$ then there exists $s \in S_2$ such that $S_1 \cup \{s\}$ is also *uwu*.

2. [**7 marks**] Adam and Eve are playing a game involving writing numbers on a blackboard. They alternate turns, starting at turn 0 with Adam playing on the even turns, and Eve playing on the odd-numbered turns. On the k^{th} turn, the player whose turn it is, writes the number $2k$ or $2k + 1$ on the board.

   A number T is fixed beforehand. The game ends as soon as the sum of the numbers on the blackboard is at least T. At this point, Adam wins if the sum is strictly greater than T, and Eve wins if the sum is exactly T.

   For which values of T does Eve have a winning strategy for this game?

3. [**7 marks**] Show that for any naturals n, k, there exists a natural number m such that we can fill an $m \times n$ grid with $1, \ldots mn$ where each number appears exactly once in the grid and the following conditions holds:

   - If the number in cell (i, j) is $a_{i,j}$ then

   $$\sum_{i=1}^{m} a_{ij_1}^{\ell} = \sum_{i=1}^{m} a_{ij_2}^{\ell} \text{ for all } 1 \leq j_1, j_2 \leq n \text{ and } 1 \leq \ell \leq k$$

   In simpler terms, the sum of the ℓ^{th} power of the elements in the j_1^{th} column equals the sum of the ℓ^{th} power of the elements in the j_2^{th} column.

4. [**7 marks**] For which natural numbers $N \geq 4$ does there exist a finite sequence of distinct natural numbers $a_1, a_2, \ldots, a_N$ such that for each index i, the condition $a_i \mid (a_{i-1} + a_{i+2})$ holds?
   (Note : all indices are taken modulo N)

5. [**7 marks**] The numbers $1, 2, \cdots, n$ (where n is an odd number) are initially written on John's computer screen. In a move, John can do one of the following tasks:

   - Square all the integers listed on the screen.
   - Pick an integer m and add it to every number on the screen.

   For instance, if $\{1, 2, 3, 4, 5\}$ were displayed initially on the screen, John can either update the numbers to $\{1, 4, 9, 16, 25\}$ (by squaring each number) or choose a number, say 6, and display $\{7, 8, 9, 10, 11\}$ on the screen, deleting the previous numbers.

Prove that he can make all the numbers displayed on the screen divisible by n in $O(\sqrt{n})$ moves.

END OF QUESTION PAPER