TESSELLATE PRESENTS

STEMS

Scholastic Test of Excellence in Mathematical Sciences

Physics Category C

Exam Date : 7th January, 2023
Exam Timing : 9 AM - 12 PM IST

Rules and Regulations

Marking Scheme

1. Time duration is **3 hours: 9 AM - 12 PM IST**. You have **20 minutes** to scan and upload your papers after that.
2. This paper contains **10 Objective** and **3 Subjective** questions. The maximum score one can obtain is **100 points**. Each objective question is worth **4 points**, and each subjective question is worth **20 points**. There is no negative marking.
3. **The subjective part will be graded only if you score above a certain cut-off (to be decided later) in the objective section of the paper.**
4. **The final cut-off shall be based on your total score (Objective + Subjective).**

Miscellaneous

1. Submit your answers through `https://forms.gle/2QMWFi98iQWKDCnr9`.
2. Write your solutions to the subjective problems neatly, then scan and generate a PDF, which you must submit through the same form. Solutions should be brief and should contain all the necessary details. Name your file as **STEMS_Physics_Roll number.pdf**, for example, **STEMS_Physics_6900.pdf**.
3. Use a good application to scan handwritten text into PDF. Kindly make sure that the answers are legible and that your furniture or flooring is not a part of the submission.
4. Make sure your PDF has a size below 10 MB.
5. Ambiguous or illegible answers will not gain credits. If you strike something out, strike it out properly so that it is clear to the evaluator what you want to read. Please avoid overwriting your answers.
6. Answers should be your own and should reflect your independent thinking process.
7. Do **NOT** post the questions on any forums or discussion groups. It will result in immediate disqualification of involved candidates when caught.
8. Sharing/discussion aimed towards solving or distribution of problems appearing in the contest while the contest is live in any kind of online platform/forum shall be considered as a failure in complying with the regulations.
9. Any form of plagiarism or failure to comply with aforementioned regulations may lead to disqualification.
10. $\hbar$ denotes the reduced Planck's constant $(h/2\pi)$, k_B denotes Boltzmann's constant, and SI units are used throughout.

Contact details

- For subject related queries, clearly mention your **name** and **category (A)** in the mail or WhatsApp text.
- For **subject related** queries, contact **stemsphysics2023@gmail.com**. Below are two more contacts, but **please use these only if the previous one is down.**
  - Adhvik Jagannathan: **adhvik@cmi.ac.in**
  - Anand Balivada: **anandb@cmi.ac.in**
- For **technical** queries, contact **tessellate.cmi@gmail.com**. Below are two more contacts, but **please use these only if the previous one is down.**
  - Siddhant Shah: **siddhants@cmi.ac.in**
  - Rohan Goyal: **rohang@cmi.ac.in**
- Your should fill in your answers to the objective questions in the google form we sent along with this question paper. Upload the scanned PDF containing the answers to subjective questions with the forementioned file name to the same form (that contained the drive link for the question paper) along with your **name, subject, category and your registered email ID** on it. Submissions by emails will be accepted only till **12:20 PM IST.**

Questions

Objective

1. Let A be a physical observable. Let $\mathcal{U}_A := \{U_\alpha\}_{\alpha\in\mathbb{R}}$ be a family of quantum operators which satisfy:
   - $U_\alpha U_\beta = U_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{R}$.
   - $U_{\delta\alpha} = 1 - \dfrac{i\delta\alpha}{\hbar} A + O(\delta\alpha^2)$ for small $\delta\alpha$.

   Which of the following statements hold?

   (a) U_α has eigenvalues $\exp\left(\dfrac{-a_k^2\alpha^2}{\hbar^2}\right)$, where a_k are eigenvalues of A.

   (b) $\mathcal{U}_A$ is a family of operators of which finitely many commute with A.

   (c) If A is the momentum operator and $|x\rangle$ is the position ket with coordinate x, $U_\alpha|x\rangle = |x+\alpha\rangle$.

   (d) If A and B are two observables, all members of $\mathcal{U}_A$ and $\mathcal{U}_B$ need not commute with each other.

2. A one dimensional particle of mass m is enclosed in a box of width L, whose center is at the origin. It currently has the wavefunction:

$$\psi(x, t=0) = \frac{1}{\sqrt{L}} \sin\left[\frac{\pi}{L}\left(x + \frac{L}{2}\right)\right]\left[1 + 2\cos\left[\frac{\pi}{L}\left(x + \frac{L}{2}\right)\right]\right]$$

   Its wavefunction after a time interval of $\dfrac{4mL^2}{\pi\hbar}$ is:

   (a) $\dfrac{1}{\sqrt{L}} \sin\left[\dfrac{\pi}{L}\left(x + \dfrac{L}{2}\right)\right]\left[-1 + 2\cos\left[\dfrac{\pi}{L}\left(x + \dfrac{L}{2}\right)\right]\right]$.

   (b) $\dfrac{1}{\sqrt{L}} \cos\left[\dfrac{\pi}{L}\left(x + \dfrac{L}{2}\right)\right]\left[1 - 2\cos\left[\dfrac{\pi}{L}\left(x + \dfrac{L}{2}\right)\right]\right]$.

   (c) $\dfrac{1}{\sqrt{L}} \sin\left[\dfrac{\pi}{L}\left(x + \dfrac{L}{2}\right)\right]\left[1 + 2\cos\left[\dfrac{\pi}{L}\left(x + \dfrac{L}{2}\right)\right]\right]$.

   (d) $\dfrac{1}{\sqrt{L}} \cos\left[\dfrac{\pi}{L}\left(x + \dfrac{L}{2}\right)\right]\left[-1 + 2\sin\left[\dfrac{\pi}{L}\left(x + \dfrac{L}{2}\right)\right]\right]$.

3. Consider a particle confined to a line, in a potential

$$U(x) = x^4$$

Given $0 < a << 1$, **a good approximation to the time period of oscillations executed by the particle between $-a$ and a is:**

(a) $\frac{\pi}{\sqrt{3}}$.

(b) $\frac{2.62206}{\sqrt{2}a}$.

(c) $\frac{1.31103}{\sqrt{2}a}$.

(d) $\frac{\pi}{\sqrt{6}}$.

4. A free particle of mass m has a wavefunction that can be written as a wavepacket, sharply peaked about some wavenumber k_0, with width $(\Delta x)_0$. **The best estimate for the amount of time required for the wavepacket to double in width is:**

(a) $\frac{2\sqrt{3}(\Delta x_0)^2 m}{\hbar}$.

(b) $\sqrt{\frac{(\Delta x_0)m}{k_0\hbar}}$.

(c) $\frac{2m}{\hbar k_0^2}$.

(d) $\frac{2m(\Delta x_0)^2}{\hbar}$

5. Consider a Helmholtz coil of radius a with the distance of separation a (two circular coils of radius a, with a common axis, separated by distance a). Take the axis of the Helmholtz coil to be the z axis, the lower coil at $z = 0$ and upper coil at $z = a$.

With $B_z(z)$ denoting the magnetic field along the axis, **which of the following statements are true?**

(i) If $|z - a/2| < a/10$, $|\frac{B_z(z)-B_z(a/2)}{B_z(a/2)}| \leq 1.2 \cdot 10^{-4}$.

(ii) $|\frac{B_z(z)-B_z(a/2)}{B_z(a/2)}| > 2/25$.

(a) (i).

(b) (ii).

(c) (i) and (ii).

(d) Neither.

6. To a strand of DNA, we can associate the "linking number" L, which is the number of times one edge of the ribbon is linked with the other edge, and is necessarily an integer. The linking number is a constant, which changes only when the strand is cut, twisted and rejoined in a suitable way.

   With L_r denoting the linking number of a relaxed DNA molecule, the energy of a molecule with linking number L is ϵi^2, with $i = L - L_r$, and some constant ϵ. **For a large collection of DNA molecules in thermal equilibrium, the average energy of a molecule is:**

   (a) $3k_BT/2$.

   (b) $5k_BT/2$.

   (c) ϵL_r^2.

   (d) $k_BT/2$.

7. You are given these statements:

   (i) Suppose there is a fixed charge Q with a charge $-q$ under its influence. The vector $\mathbf{p}\times\mathbf{L} - \dfrac{mQq}{4\pi\epsilon_0 r}\mathbf{r}$ is conserved, where $\mathbf{p}, \mathbf{L}$ are the momentum and angular momentum of $-q$ about Q.

   (ii) If the electric potential is known for all points in space satisfying $x^4 + 2y^4 + 8y^2 + 13z^4 = 2023$, then the potential is determined for all points satisfying $x^4 + 2y^4 + 8y^2 + 13z^4 \leq 2023$, provided there's no charge in this region.

   Which of these is/are true?

   (a) (i).

   (b) (ii).

   (c) (i) and (ii).

   (d) None.

8. One can represent ensembles in quantum mechanics using an operator known as the density operator, defined by:

$$\rho = \sum_i w_i |\alpha_i\rangle\langle\alpha_i|$$

   Here $|\alpha_i\rangle$ represent the possible states of each particle in the ensemble, and w_i the probability of a particle being in that state.

   An ensemble consisting of two state particles is characterized by the density operator $\rho = 0.5\,|+\rangle\langle+| + 0.1\,|+\rangle\langle-| + 0.1\,|-\rangle\langle+| + 0.5\,|-\rangle\langle-|$. **Calculate the average value of $S_x = (|+\rangle\langle-| + |-\rangle\langle+|)$ in this ensemble.**

   (a) 0.2

   (b) 0.1

   (c) 0.57

   (d) 0.14

Questions 9 and 10 are based on the paragraph below.

Deutereum and Tritium undergo nuclear fusion to produce Helium-4 and a free neutron. A large fraction of the energy generated in this reaction is attributed to the neutron, giving it a large speed v in a certain frame S. Free neutrons have a mean lifetime of τ in rest frame. Assume units where $c = 1$.

9. **Which of the following holds?**

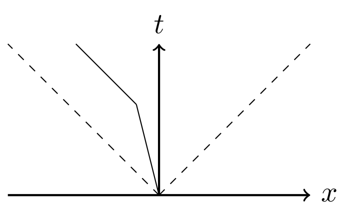
   (a) Observers in frame S observe that the mean distance a neutron travels is τv.

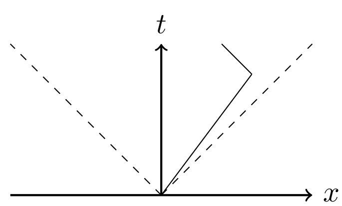
   (b) Observers in frame S observe that the number of neutrons to survive after a distance d is $N \exp\left(\dfrac{d}{v\tau}\right)$.

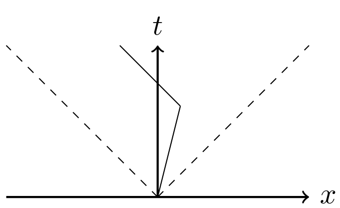
   (c) Observers with the same velocity as the neutron beam observe the mean lifetime of the neutron to be $\dfrac{\tau}{\sqrt{1 - v^2}}$.

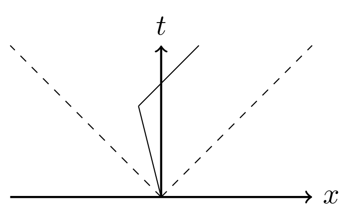
   (d) Observers in frame S observe that the number of neutrons to survive after a distance d is $N \exp\left(\dfrac{d\sqrt{1 - v^2}}{v\tau}\right)$.

10. Suppose among these neutrons is a special neutron, which we shall call the golden neutron. The golden neutron always decays immediately after the mean lifetime it observes itself to have. Furthermore, when it does decay, it releases a light pulse in the opposite direction. **Which of these spacetime diagrams best represent how an observer moving at the speed 0.5 units along the neutron beam observes this phenomenon?**

(a) 

(b) 

(c) 

(d) 

Subjective

1. A system with two degrees of freedom is described by the Hamiltonian given below:

$$H(q_i, p_i) = \frac{1}{2m}\left[q_1^2 + 2q_2 \sin^2 p_2 (1 + \sqrt{q_2}\cos p_2)^2\right] + \frac{k}{2}\left[p_1^2 + 28 q_2 (1 + \sqrt{q_2}\cos p_2)^2\right]$$

k is given to be a positive constant.

   (a) If the initial positions and momenta of the system are given by $p_1(0) = \frac{4}{7}$, $q_1(0) = 0 = p_2(0)$, $q_2(0) = 1$, **describe the implicit solutions of the system**, i.e. determine the functions f_i, g_i for $i = 1, 2$ such that $f_i(q_i) = g_i(t)$. Here f_i is a real valued function whose domain is the set of possible values q_i can take.

   (b) **Calculate the value of** I, where

$$I = \frac{1}{(2\pi\hbar)^4}\int_{H\le 1} dq_1 dq_2 dp_1 dp_2$$

2. Consider a system of N identical bosons confined in a chamber, such that the energy states they can occupy are E_k, for $k \in \mathbb{N}$. These bosons are special in the sense that they occupy energy levels in pairs, i.e. if n_k is the number of bosons occupying energy E_k, then n_k is even for all k. These bosons are at temperature $\frac{1}{k_B\beta}$ and chemical potential μ.

   (a) **Find the thermodynamic grand potential Ω of this system, and thus find the average values of its occupation numbers n_k.**

   (b) Suppose the energy levels of this chamber are given by $E_k = \mu + \ln\left[\frac{2k^2}{N\beta} + 1\right]$. **Find the value of β for which this is possible.**

3. Consider a solenoid whose axis is along the $\hat{\mathbf{z}}$ direction, with a radius R and magnetic flux ϕ, with the field confined to the solenoid. Now, consider a plane wave electron beam travelling along the -$\hat{\mathbf{x}}$ direction with wavenumber k.

   (a) [**15 marks**] Working in cylindrical coordinates, **derive an equation for the wavefunction of the electron beam for** $r > R$.

   (b) [**5 marks**] The solution to the equation derived in (a) behaves asymptotically as:

$$\psi \approx e^{-i(\alpha\theta + kr\cos\theta)} + \frac{e^{ikr}}{\sqrt{2\pi i k r}}\sin(\pi\alpha)e^{-i\theta/2}\sec(\theta/2)$$

   Calculate the probability of finding the scattered electron along the y axis.

All the Best!!