TESSELLATE PRESENTS

STEMS

Scholastic Test of Excellence in Mathematical Sciences

Subject Category C

Exam Date : 8th January, 2023
Exam Timing : 11:00 AM IST - 5:00 PM IST

Rules and Regulations

Marking Scheme

1. The question paper contains **sixteen** questions, **two** objective question, **seven** integer type questions (**Part A**), and **seven** subjective questions (**Part B**).
2. Each subjective question is worth **7 marks**. The marks for integer type questions and objective questions are indicated against the respective question numbers in the question paper.
3. A candidate's submission for **Part B** of the exam will be graded only if the total score obtained by the candidate in **Part A** is at least 12 **OR** the candidate is among the top 10 candidates in **Part A**.
4. Time duration is **6 hours: 11:00 AM IST - 5:00 PM IST**.
   Submit your answers on the google form given below by **5:30 PM IST**.

Miscellaneous

1. Use the google form: **https://forms.gle/YTtbcVk8LyVEjBGn6** , to submit your answers.
2. For **Part A**, give answers in the form of a single integer, without any whitespaces, commas, periods, semi-colons, underscores or any other special characters. Submissions with special characters such as these will NOT be graded (hyphens are allowed for negative integers).
3. For **Part B**, you can either LaTeX or handwrite your solutions neatly.
   Submit a PDF file (either scanned or LaTeXed) **ONLY**. No other form of file submission will be accepted. Name your file **"math_rollnumber"** (here rollnumber is the 4 digit schoolpay/airpay receipt number generated at the time of registration).
4. Make sure to keep the file size below the 10 MB limit. You can use online file compression services in case your file size exceeds 10 MB.
5. Use a good application to scan handwritten text into PDF. Kindly make sure that the answers are legible and that your furniture or flooring is not a part of the submission.
6. Solutions should be brief and should contain all the necessary details. Ambiguous or illegible answers will not gain credits. If you strike something out, strike it out properly so that it is clear to the evaluator what you want to be read. Please avoid overwriting your answers.
7. Do **NOT** post/share the questions appearing in the contest on any forums or discussion groups while the contest is live. It will result in immediate disqualification of involved candidates when caught.
8. Answers should be your own and should reflect your independent thinking process. Any form of plagiarism or failure to comply with aforementioned regulations may lead to disqualification.

Contact details

- For subject related queries, clearly mention your **category (A/B/C)** in the mail or WhatsApp text.
- For **subject related** queries, contact:
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Note: Use the personal emails only if the official email is unreachable. Use WhatsApp only if absolutely necessary, otherwise email is preferred.

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Questions

Part A

1. **[2 marks]** Let $n \in \mathbb{N}_{\geqslant 2}$. Let $X := \{M \in M_n(K) \mid \forall A, B \in M_n(K) \ \ (AB = M \ \Rightarrow \ BA = M)\}$. Which one of these statements regarding X is correct?

   (A) X forms a group under matrix multiplication and is isomorphic to $(K^*)^n$.

   (B) X forms a group under matrix multiplication and is isomorphic to K^*.

   (C) X forms a group under matrix multiplication and is isomorphic to $\mathrm{GL}_n(K)$.

   (D) X does not form a group under matrix multiplication.

2. **[2 marks]** If L is a 14×14 matrix such that its characteristic polynomial f is also the minimal polynomial of some algebraic number α, then what is the value of the least natural number n, such that $L^n = I$, where I is the identity matrix?

   (A) 14

   (B) 28

   (C) 42

   (D) None of the above

3. **[3 marks]** Let $p \geq 5$ be a prime number. Let $S_p := \left\{a \in \mathbb{N} : 1 \leq a \leq p-1, \left(\frac{a}{p}\right) = 1 = \left(\frac{a+3}{p}\right)\right\}$.

   Find $|S_{2027}|$. (Here $\left(\frac{a}{p}\right)$ denotes the Legendre Symbol.)

4. **[2 marks]** The following 100 numbers are written on the board:

   $$2^1 - 1, 2^2 - 1, 2^3 - 1, \cdots, 2^{100} - 1$$

   Alice chooses two numbers a, b, erases them and writes the number $\dfrac{ab-1}{a+b+2}$ on the board. She keeps doing this until a single number remains on the board.

   If the sum of all possible numbers she can write on the board is p/q where p, q are coprime, then what is the value of $\log_2(p+q)$?

5. **[2 marks]** Consider the set S of permutations of $1, 2, \cdots 2022$ such that for all numbers k in the permutation, the number of numbers less than k that follow k is even.

   For example, for $n = 4$; $S = \{[3,4,1,2];\ [3,1,2,4];\ [1,2,3,4];\ [4,1,2,3]\}$.

   If $|S| = (a!)^b$ where $a, b \in \mathbb{N}$, then find the product ab.

6. **[2 marks]** Find the rank of the 2023×2023 matrix A, given by $A_{ij} = \cos(i-j), \forall 1 \le i, j \le 2023$.

7. **[4 marks]** For a group G, consider the equivalence relation given by $x \sim y$ iff $\exists\ f \in \mathrm{Aut}(G)$ s.t. $f(x) = y$. Denote $\alpha(G)$ to be the no. of equivalence classes of G under the relation and set $\varphi(G) = \dfrac{\alpha(\mathrm{G})}{|G|}$.

   (a) **[2 marks]** How many finite cyclic groups G (upto isomorphism) are there such that $\varphi(G) \ge \dfrac{5}{8}$, and the order of G is a prime power?

   (b) **[2 marks]** How many finite abelian groups G (upto isomorphism) are there such that $\varphi(G) \ge \dfrac{5}{8}$?

8. **[3 marks]** Let

$$S := \{f \mid f \text{ is a } C^{2023}\text{function on } [0,1] \text{ such that } f^{(2023)} = f\}$$

   Here $f^{(2023)}$ is the 2023rd derivative of f. Given that $f(1/n) = 0\ \forall\ n \in \mathbb{N}$, what is the value of $f(1/\pi)$?

9. **[2 marks]** Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integer matrix and let $\mathcal{C}$ denote the unit circle in the complex plane.

$$\mathcal{C} = \{z \in \mathbb{C} \mid \|z\| = 1\}$$

   Consider the action of this matrix on the complex plane, which takes a point α on the plane, and sends it to the point $\dfrac{a\alpha + b}{c\alpha + d}$.
   Let S denote the sum of all 2×2 matrices M with integer entries and determinant 1, such that M maps $\mathcal{C}$ into $\mathcal{C}$. Find the sum of the entries of S.

Part B

1. **[7 marks]** Let X be a countable set. Does there exists a family $\mathcal{A}$ of subsets of X such that the following hold?

   - $\mathcal{A}$ is uncountable.
   - For all distinct elements $P, Q \in \mathcal{A}$, $P \cap Q$ is finite.

2. **[7 marks]** Define a positive integer n to be a *fake square* if either $n = 1$ or n can be written as a product of an even number of not necessarily distinct primes. Prove that for any even integer $k \geq 2$, there exist distinct positive integers $a_1, a_2, \ldots, a_k$ such that the polynomial $(x+a_1)(x+a_2)\cdots(x+a_k)$ takes 'fake square' values for all $x = 1, 2, \ldots, 2023$.

3. **[7 marks]** Define an equivalence relation R on $\mathbb{R}^{\mathbb{R}}$ as

$$R := (f \sim g) \iff (\exists\, \alpha \in \mathbb{R} \mid f(x) - g(x) \text{ is continuous at all } x > \alpha)$$

   Find the cardinality of the set of all equivalence classes.

4. **[7 marks]** Let H be an abelian group and G be a subgroup of H. Show that any group homomorphism σ from G to $\mathbb{Q}$, the abelian group of rational numbers, extends to a group homomorphism from H to $\mathbb{Q}$.

5. **[7 marks]** Show that 25 can be written as a sum of 4 cubes in infinitely many ways.

6. **[7 marks]** Let $n > 1$ be an integer and let A be a $n \times n$ **non-identity** ($A \neq I$) matrix with rational entries ($A \in M_n(\mathbb{Q})$). Let $p > n + 1$ be a prime number. Prove that,

$$A^p + A^{p-1} + \cdots + A \neq pI$$

7. **[7 marks]** Let $K_{n,n}$ denote the bipartite graph with two sets of S_1, S_2 of n vertices each, such that each vertex of S_1 is adjacent to each vertex of S_2. For a positive integer n, let $f(n)$ denote the largest positive integer satisfying the following property:

   Given any colouring of the edges of $K_{n,n}$ into two colours, red and blue, we can always find a subgraph of $K_{n,n}$, isomorphic to $K_{f(n),f(n)}$ such that all its edges are of the same colour.

   Determine whether the following statement is true or not:

   For each positive integer m, we can find a natural number N, such that for any integer $n \geq N$, $f(n) \geq m$.

END OF QUESTION PAPER