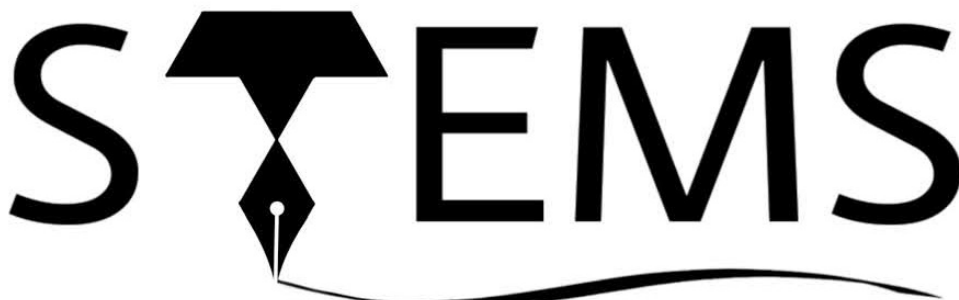




TESSELLATE PRESENTS



Scholastic Test of Excellence in Mathematical Sciences

Subject Category B

Exam Date : 14th January, 2023
Exam Timing : 10:00 AM IST - 1:00 PM IST



Rules and Regulations

Marking Scheme

1. The question paper contains **five** subjective problems.
2. Each subjective question is worth **7 marks**.
3. Time duration is **3 hours: 10:00 AM IST - 1:00 PM IST**.
Submit your answers on the google form given below by **1:30 PM**.

Miscellaneous

1. Use the google form: <https://forms.gle/k9L8UfAp5cwvfV9Q9> , to submit your answers.
2. You can either LaTeX or handwrite your solutions neatly.
Submit a PDF file (either scanned or LaTeXed) **ONLY**. No other form of file submission will be accepted. Name your file “**math_rollnumber**” (here rollnumber is the 4 digit schoolpay/airpay receipt number generated at the time of registration).
3. Make sure to keep the file size below the 10 MB limit. You can use online file compression services in case your file size exceeds 10 MB.
4. Use a good application to scan handwritten text into PDF. Kindly make sure that the answers are legible and that your furniture or flooring is not a part of the submission.
5. Solutions should be brief and should contain all the necessary details. Ambiguous or illegible answers will not gain credits. If you strike something out, strike it out properly so that it is clear to the evaluator what you want to be read. Please avoid overwriting your answers.
6. Do **NOT** post/share the questions appearing in the contest on any forums or discussion groups while the contest is live. It will result in immediate disqualification of involved candidates when caught.
7. Answers should be your own and should reflect your independent thinking process. Any form of plagiarism or failure to comply with aforementioned regulations may lead to disqualification.

Contact details

- For subject related queries, clearly mention your **category (A/B/C)** in the mail.
- For **subject related** queries, contact:
 - Official email ID: **stemsmath2023@gmail.com**
 - Rishabh Sharma
Email : **rishabh@cmi.ac.in**
 - Ananya Ranade
Email: **ananyar@cmi.ac.in**

Note: Use the personal emails only if the official email is unreachable.



Questions

1. [7 marks] There are 100 piles of stones, with 100 stones in each pile. In a move, we can select some 50 piles, remove 1 stone from each of them, and add 1 stone each to the remaining 50 piles. The number of stones in a pile is always > 0 after any move, i.e. no pile can be empty. Applying these moves, is it possible to obtain a position such that the number of stones in the piles is strictly increasing from left to right?

2. [7 marks] A set of real numbers A is said to have Intermediate Value Property (IVP) iff

$$(a, b \in A, a < b) \implies (\exists c \in A \mid a < c < b)$$

Given that S has IVP, evaluate the validity of the following statements.

- (i) S countable and bounded \implies there necessarily exists some interval (a, b) such that S is dense in (a, b) .
- (ii) S uncountable and bounded \implies there necessarily exists some interval (a, b) such that S is dense in (a, b) .
- (iii) S countable and unbounded \implies there necessarily exists some interval (a, b) such that S is dense in (a, b) .
- (iv) S uncountable and unbounded \implies there necessarily exists some interval (a, b) such that S is dense in (a, b) .

Note: A set $S \subseteq \mathbb{R}$ is said to be **dense** in another set $T \subseteq \mathbb{R}$ iff for every $x \in T$ and every $\epsilon > 0$, there is an element $y \in S$ such that $|x - y| < \epsilon$. For example, \mathbb{Q} is dense in \mathbb{R} .

3. [7 marks] Find all pairs (k, n) of natural numbers for which there exists integers $1 < m_1 < \dots < m_k$ such that $m_1! + m_2! + \dots + m_k! = 2^n$.
4. [7 marks] Let ABC be a triangle, D, E, F be the midpoints of BC, CA, AB respectively and P, Q be the feet of the perpendiculars from B, C to AC, AB respectively. Assume that the circumcircle of AFD cuts DP again at G and the circumcircle of AED cuts DQ again at H . Let $X = EH \cap FG$ and $T = AD \cap \odot(DEF)$. Prove that $DX = DT$.
5. [7 marks] Consider a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that if $x, y \in \mathbb{N}$ such that $x < y$ and $x \mid y$ then $\exists k \in \mathbb{N}$ such that $f^k(x) = y$ and $x \mid k$. Prove that the set of functions satisfying this property has cardinality the same as that of \mathbb{R} .

END OF QUESTION PAPER