



TESSELLATE PRESENTS



Scholastic Test of Excellence in Mathematical Sciences

Subject Category A

Exam Date : 8th January, 2023
Exam Timing : 11:00 AM IST - 5:00 PM IST



Rules and Regulations

Marking Scheme

1. The question paper contains **fourteen** questions, **eight** integer type questions (**Part A**), and **six** subjective questions (**Part B**).
2. Each subjective question is worth **7 marks**. The marks for integer type are indicated against the respective question numbers in the question paper.
3. A candidate's submission for the **Part B** of the exam will be checked only if the total score obtained by the candidate in the **Part A** is at least 9 **OR** they are in the top 20 candidates for **Part A**.
4. Time duration is **6 hours: 11:00 AM IST - 5:00 PM IST**.
Submit your answers on the google form given below by **5:30 PM IST**.

Miscellaneous

1. Use the google form: <https://forms.gle/KPCJ7dTHTqeCid729> , to submit your answers.
2. For **Part A**, give answers in the form of a single integer, without any whitespaces, commas, periods, semi-colons, underscores or any other special characters. Submissions with special characters such as these will NOT be graded (hyphens are allowed for negative integers).
3. For **Part B**, you can either LaTeX or handwrite your solutions neatly.
Submit a PDF file (either scanned or LaTeXed) **ONLY**. No other form of file submission will be accepted. Name your file "**math_rollnumber**" (here rollnumber is the 4 digit schoolpay/airpay receipt number generated at the time of registration).
4. Make sure to keep the file size below the 10 MB limit. You can use online file compression services in case your file size exceeds 10 MB.
5. Use a good application to scan handwritten text into PDF. Kindly make sure that the answers are legible and that your furniture or flooring is not a part of the submission.
6. Solutions should be brief and should contain all the necessary details. Ambiguous or illegible answers will not gain credits. If you strike something out, strike it out properly so that it is clear to the evaluator what you want to be read. Please avoid overwriting your answers.
7. Do **NOT** post/share the questions appearing in the contest on any forums or discussion groups while the contest is live. It will result in immediate disqualification of involved candidates when caught.
8. Answers should be your own and should reflect your independent thinking process. Any form of plagiarism or failure to comply with aforementioned regulations may lead to disqualification.

Contact details

- For subject related queries, clearly mention your **category (A/B/C)** in the mail or WhatsApp text.
- For **subject related** queries, contact:
 - Official email ID: stemsmath2023@gmail.com



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Note: Use the personal emails only if the official email is unreachable. Use WhatsApp only if absolutely necessary, otherwise email is preferred.

- For **technical** queries, contact:
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Questions

Part A

1. [2 marks] The following 100 numbers are written on the board:

$$2^1 - 1, 2^2 - 1, 2^3 - 1, \dots, 2^{100} - 1$$

Alice chooses two numbers a, b , erases them and writes the number $\frac{ab-1}{a+b+2}$ on the board. She keeps doing this until a single number remains on the board.

If the sum of all possible numbers she can end up with is p/q where p, q are coprime, then what is the value of $\log_2(p+q)$?

2. [2 marks] Consider the set S of permutations of $1, 2, \dots, 2022$ such that for all numbers k in the permutation, the number of numbers less than k that follow k is even.

For example, for $n = 4$; $S = \{[3, 4, 1, 2]; [3, 1, 2, 4]; [1, 2, 3, 4]; [4, 1, 2, 3]\}$.

If $|S| = (a!)^b$ where $a, b \in \mathbb{N}$, then find the product ab .

3. [3 marks] Given a triangle ABC with angles $\angle A = 60^\circ$, $\angle B = 75^\circ$, $\angle C = 45^\circ$, let H be its orthocentre, and O be its circumcenter. Let F be the midpoint of side AB , and Q be the foot of the perpendicular from B onto AC . Denote by X the intersection point of the lines FH and QO . Suppose the ratio of the length of FX and the circumradius of the triangle is given by $\frac{a+b\sqrt{c}}{d}$, then find the value of $1000a + 100b + 10c + d$.

4. [3 marks] Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that the following conditions hold:

- (1) $f(1) = 1$.
- (2) $(x+y)/2 < f(x+y) \leq f(x) + f(y) \quad \forall x, y \in \mathbb{N}$.
- (3) $f(4n+1) < 2f(2n+1) \quad \forall n \geq 0$.
- (4) $f(4n+3) \leq 2f(2n+1) \quad \forall n \geq 0$.

Find the sum of all possible values of $f(2023)$.

5. [2 marks] Consider a polynomial $P(x) \in \mathbb{R}[x]$, with degree 2023, such that $P(\sin^2(x)) + P(\cos^2(x)) = 1$ for all $x \in \mathbb{R}$. If the sum of all roots of P is equal to p/q with p, q coprime, then what is the product pq ?
6. [2 marks] There are 5 vertices labelled $1, 2, 3, 4, 5$. For any two pairs of vertices u, v , the edge uv is drawn with probability $1/2$. If the probability that the resulting graph is a tree is given by p/q



where p, q are coprime, then find the value of $q^{1/9} + p$.

7. [**1 mark**] Suppose a biased coin gives head with probability $2/3$. The coin is tossed repeatedly, if it shows heads then player A rolls a fair die, otherwise player B rolls the same die. The process ends when one of the players get a 6, and that player is declared the winner.

If the probability that A will win is given by m/n where m, n are coprime, then what is the value of m^2n ?

8. [**2 marks**] For how many pairs of primes (p, q) , is $p^2 + 2pq + 1$ also a prime?



Part B

- [7 marks] If in $\triangle ABC$, $AC = 15$, $BC = 13$ and $IG \parallel AB$ where I is the incentre and G is the centroid, what is the area of the triangle ABC ?
- [7 marks] Given a complete bipartite graph on n, n vertices (call this $K_{n,n}$), we colour all its edges using two colours, red and blue. What is the **least** value of n such that for any colouring of the edges of this graph, there will exist at least one monochromatic 4-cycle?
- [7 marks] Suppose f is a non constant polynomial with integer coefficients with the following property:
 - $f(0)$ and $f(1)$ are both odd.
 - Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, with,

$$a_1 = f(1) + 1, \quad a_n = f(1)f(2)f(3) \dots f(n) + 1$$

Then prove that the set of primes dividing at least one term of the sequence is infinite.

- [7 marks] Alice has $n > 1$ one variable quadratic polynomials written on paper she keeps secret from Bob. On each move, Bob announces a real number and Alice tells him the value of one of her polynomials at this number. Prove that there exists a constant $C > 0$ such that after $C \times n^5$ questions, Bob can determine one of Alice's polynomials.
- [7 marks] A convex quadrilateral $ABCD$ is such that $\angle B = \angle D$ and are both acute angles. E is on AB such that $CB = CE$ and F is on AD such that $CF = CD$. If the circumcenter of CEF is O_1 and the circumcenter of ABD is O_2 . Prove that C, O_1, O_2 are collinear.
- [7 marks] Define a positive integer n to be a *fake square* if either $n = 1$ or n can be written as a product of an even number of not necessarily distinct primes. Prove that for any even integer $k \geq 2$, there exist distinct positive integers a_1, a_2, \dots, a_k such that the polynomial $(x+a_1)(x+a_2) \dots (x+a_k)$ takes 'fake square' values for all $x = 1, 2, \dots, 2023$.

END OF QUESTION PAPER