TESSELLATE PRESENTS

STEMS

Scholastic Test of Excellence in Mathematical Sciences

Computer Science Category A

Exam Date	**:**	**7th Janurary 2023**
Exam Timing	**:**	**3 PM IST - 6 PM IST**
Form Link	**:**	**https://forms.gle/jkKZoaSePVZ4AQgw6**

Rules and Regulations

Marking Scheme

1. Time duration for the exam is **3 hours: 3 PM - 6PM**. Submit your answers(i.e.- the Google form: https://forms.gle/jkKZoaSePVZ4AQgw6) by **6:30 PM**. **This is a hard deadline.** No submissions past 6:30 PM will be accepted. Submissions should only be via form and any submission via mail will be ignored.
2. The marks for each question is mentioned in bold square brackets before its statement.
3. **The subjective part will be graded only if you score above a certain cut-off (to be decided later) in the objective section of the paper.**
4. **The final cut-off shall be based on your total score (Objective + Subjective).**

Miscellaneous

1. Write your solutions neatly, and submit the scanned PDF. Solutions should be brief and should contain all the necessary details. Name your file as **"Full name Subject Name Cat A"**. Note that Category B consists of students from the first year of university to the final year. If you do not belong to this class of participants, please do NOT attempt this paper.
2. **Please note that the form DOES NOT auto-submit.** You have to press the submit button yourself.
3. Use a good application to scan handwritten text into PDF. Kindly make sure that the answers are legible and that your furniture or flooring is not a part of the submission.
4. Ambiguous or illegible answers will not gain credits. If you strike something out, strike it out properly so that it is clear to the evaluator what you want to read. Please avoid overwriting your answers.
5. Answers should be your own and should reflect your independent thinking process.
6. Do **NOT** post the questions on any forums or discussion groups. It will result in immediate disqualification of involved candidates when caught.
7. Sharing/discussion aimed towards solving or distribution of problems appearing in the *contest while the contest is live in any kind of online platform/forum shall be considered as a failure in complying with the regulations.*
8. Any form of plagiarism or failure to comply with aforementioned regulations may lead to disqualification.
9. You may want to refer to Big O Notation if you are not aware of what it is.

Contact details

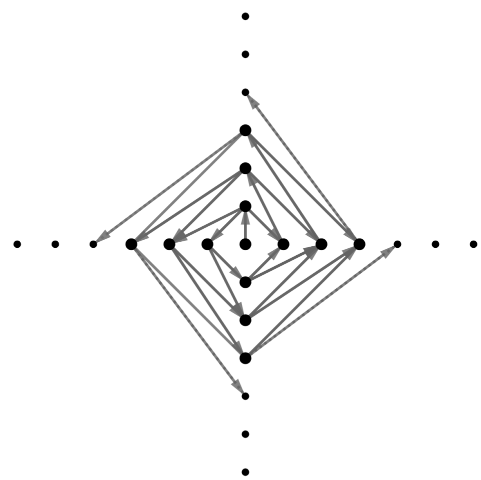
- For subject related queries, clearly mention your **category (A)** in the mail.
- For **subject related** queries, contact:
  - Official email ID: **stemscsenquiry@gmail.com**(This email will be actively checked during the exam. It is advisable to send queries to this email address.)
  - Rajdeep Ghosh: rajdeep@cmi.ac.in
  - Rohan Goyal: rohang@cmi.ac.in
- For **technical** queries, contact:
  - Siddhant Shah : siddhants@cmi.ac.in
  - Rohan Goyal : rohang@cmi.ac.in

Questions

Objective

1. **[2 marks]** Let M the number of subsets of $[12000] = \{1, 2, \ldots, 12000\}$ such that for any two elements of the subset, the absolute value of the difference between them is a multiple of 3 or 4. Find $M (\text{mod } 1000)$

2. **[3 marks]** Consider the following network:



The vertices are numbered as follows: The vertex at the center is numbered 0. The vertices in upper vertical line are numbered $1, 5, 9, \ldots$, those in the left horizontal line are numbered $2, 6, 8 \ldots$, those in the lower vertical line are numbered $3, 7, 11, \ldots$ and those in the right horizontal line are numbered $4, 8, 12 \ldots$.
You can only travel along the indicated lines from a lower number to a higher number. Find the number of paths from 0 to 21.

3. **[2 marks]** Consider the following three functions:

$$f_1(n) = n^{n^n}$$

$$f_2(n) = (100n!)^{n!}$$

$$f_3(n) = F_n^{F_n}$$

where F_n is the n^{th} Fibonacci number. Given functions f and g, we'll write "$f < g$" if $f(n) = o(g(n))$. If you're not aware of the meaning of $o(g(n))$, jump to "Little-o notation" at Big O Notation. Then choose the correct alternative out of the following:

   (a) $f_1 < f_2 < f_3$

   (b) $f_2 < f_1 < f_3$

   (c) $f_3 < f_1 < f_2$

   (d) $f_1 < f_3 < f_2$

   (e) $f_3 < f_2 < f_1$

   (f) $f_2 < f_3 < f_1$

4. **[3 marks]** Consider the integer lattice of points $\{(x, y) \mid x, y \in \mathbb{Z}\}$. We start with three points marked on this lattice at $(0,0)$, $(1,0)$, $(0,1)$. Now, in every move, you can pick a point currently marked on the lattice say A. Let the other two points be B and C. Then you can mark any other point D on the lattice as long as $AD \parallel BC$ and you have to unmark A. Assume that after some moves, two of the points are at $(0,0)$ and $(2023,1)$. How many possibilities are there for the third point if it's coordinate is (x, y) such that $|x| < 2023$ and $|y| \leq 2023$?

5. **[3 marks]** For any function $f : [5] \mapsto [5]$, we say the function is $2-$*colourable*, if $\forall\ i, j \in [5], \exists\ k, l > 0$ such that $f^k(i) = f^l(j)$ and $\exists\ g : [5] \mapsto [2]$ such that $\forall i,\ g(i) \neq g(f(i))$. Find the number of *2-colourable* functions from $[5]$ to $[5]$. [1]

6. **[2 marks]** Let $G = K_{2022}$ be the complete graph on 2022 vertices. Now consider colourings χ on G with two colours. Find the length of the longest possible walk[2] over all possible colorings such that the endpoints of every edge in the walk have different colours.

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[1] $[5] = \{1, 2, 3, 4, 5\}$

[2] A walk can have repeat vertices but not edges

Subjective

1. **[7 marks]** There is a graph G with n vertices and e edges. A *command* is a pair (v_i, C_i). By imposing the command on G, we mean colouring all neighbours of v_i with the colour C_i (possibly overwriting a previously assigned colour). Provide an algorithm that runs in $O(e+n+q)$ time which takes as input a list of q commands and outputs the colour of every vertex of the graph obtained by imposing the commands on G in the listed order.

2. **[7 marks]** You have an array (which you know) $a_1, \ldots, a_n$. In each move, you can partition it into contiguous subsets and reverse each contiguous subset.

   - **[4 marks]** Show that the array can be sorted in $O(\log^2 n)$ moves.
   - **[3 marks]** Show that there is no deterministic algorithm that sorts each array in $o(\log n)$ moves.

3. **[7 marks]** You have n distinct reals $a_1, \ldots, a_n$ for $n > 3$. Initially, all of these numbers are *unmarked*. In each move you can compare two unmarked numbers a_i and a_j. After comparison, you *mark* one of a_i and a_j. In the end, you are left with one unmarked number, say a_k. Can you ensure that atleast $O(\sqrt{n})$ numbers in the array are greater than a_k and atleast $O(\sqrt{n})$ numbes in the array are less than a_k.

4. **[7 marks]** We consider the following variation on the classical 20 questions game: Two players Alice and Bob play a game where Alice has a set $S \subset \{0,1\}^n$ such that $0 < |S| = k < 2^{n-1}$. Now, Bob can ask Alice *questions* which are any functions $q : \{0,1\}^n \mapsto \{0,1\}$. Now, Alice can choose any $s \in S$ in response to a question and respond with $q(s)$.

   - **[3 marks]** Find the smallest l in terms of n and k such that Bob can always ask some questions and find a list L of size atmost l such that $L \cap S \neq \phi$
   - **[4 marks]** Show that Bob can find such a list with atmost n^{2k} questions.

5. **[7 marks]** Consider a long room which has been divided into $n+1$ chambers linearly (by installing n doors). You start in the first chamber. Every time you unlock a door, you are transported back to the first chamber. When you cross an unlocked door, it closes itself again. What is the least number of times you need to unlock a door to get into the last chamber?

6. **[7 marks]** Define the *execution period* of a directed graph G to be the minimum number t such that $\forall u, v$ in the graph, the probability that v lies on a random t length walk from u is atleast $\frac{1}{2}$. Note that this might not always exist.

   Show that there exists $c > 1$ such that $\forall n \geq 3$, $\exists$ a graph G_n such that the *execution period* of G_n exists is atleast c^n.

All the Best!!