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by kapilpavase, Kayak

Q1 Let $M > 1$ be a natural number. Tom and Jerry play a game. Jerry wins if he can produce a function $f : \mathbb{N} \to \mathbb{N}$ satisfying

-$f(M) \neq M$
- $f(k) < 2k$ for all $k \in \mathbb{N}$
- $f^{f(n)}(n) = n$ for all $n \in \mathbb{N}$. For each $\ell > 0$ we define $f^\ell(n) = f\left(f^{\ell-1}(n)\right)$ and $f^0(n) = n$

Tom wins otherwise. Prove that for infinitely many M, Tom wins, and for infinitely many M, Jerry wins.

Proposed by Anant Mudgal

Q2 Does there exist a nonzero algebraic number α with $|\alpha| \neq 1$ such that there exists infinitely many positive integers n for which there's $\beta_n \in \mathbb{C}$ with $\beta_n \in \mathbb{Q}(\alpha)$ and $\beta_n^n = \alpha$?

Q3 Let $p \in \mathbb{N} \setminus \{0, 1\}$ be a fixed positive integer. Prove that for every $K > 0$, there exist infinitely many n and N such that there are atleast $\frac{KN}{\log(N)}$ primes among the following N numbers given by

$$n + 1, n + 2^p, n + 3^p, \cdots, n + N^p.$$

Proposed by Bimit Mandal

Q4 Let n be a fixed positive integer.
- Show that there exist real polynomials $p_1, p_2, p_3, \cdots, p_k \in \mathbb{R}[x_1, \cdots, x_n]$ such that

$$(x_1+x_2+\cdots+x_n)^2+p_1(x_1, \cdots, x_n)^2+p_2(x_1, \cdots, x_n)^2+\cdots+p_k(x_1, \cdots, x_n)^2 = n(x_1^2+x_2^2+\cdots+x_n^2)$$

- Find the least natural number k, depending on n, such that the above polynomials $p_1, p_2, \cdots, p_k$ exist.

Q5 Find the largest constant c, such that if there are N discs in the plane such that every two of them intersect, then there must exist a point which lies in the common intersection of $cN + O(1)$ discs

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