TESSELLATE PRESENTS

STEMS

Scholastic Test of Excellence in Mathematical Sciences

Mathematics Category B

Exam Date : 26th January 2020
Exam Timing : 12noon - 6pm

Rules and Regulations

Marking Scheme

1. The question paper is divided in two parts -
   Objective (15 questions) + Subjective (8 questions, 6 of which shall be graded).
2. Each objective question is worth **2 points**.
3. Each subjective problem is worth **10 points**.
4. There is no negative marking in any section.
5. **The subjective part will be graded only if you score above a certain cut-off (to be decided later) in the objective section of the paper**.
6. **For the final score, your total score (subjective + objective) will be taken into consideration.**

Solution guidelines

1. You are **NOT** required to show your work for the objective part of the paper.
   Only tick your option choices in a tabular format drawn on a blank sheet.
   A sample is shown below.

Q. No.	(a)	(b)	(c)	(d)
1		✓		
2			✓	
3		✓		
4	✓			
5				✓

Fig. - Sample objective answer submission

2. Provide a complete solution/proof for subjective questions. The solutions must be correct, original, detailed and clearly understandable for full credit.
   Partial credit might be awarded to incomplete proofs, based on the progress made towards solving the problem.
3. Draw clear, well-labeled diagrams wherever necessary.

Miscellaneous

1. Answers should be your own and should reflect your independent thinking process.
2. Do **NOT** post the questions on any forums or discussion groups. It will result in immediate disqualification of involved candidates when caught.
3. Answers should be written clearly, in a legible way. Formal proofs are required wherever asked for. Unclear reasoning might not be awarded points, draw clear diagrams wherever necessary.
4. Sharing/discussion aimed towards solving or distribution of problems appearing in the contest while the contest is live in any kind of online platform/forum shall be considered as a failure in complying with the regulations.
5. Any form of plagiarism or failure to comply with aforementioned regulations may lead to disqualification.

Contact details - ONLY for subject related queries

- Please do not call these people for technical problems or submission inquiries. Only if you find an ambiguity in a question and need clarification, use these contacts.
- As our phone numbers will be busy, **we prefer WhatsApp & email queries** instead. Only call us if absolutely necessary.
- Try to solve all your submission related doubts from information in the next page ONLY. We have included all details in the next section.

Arka Karmakar - 9933332626	Shubham Saha - 7783020940
Query email - stems.2020.enq.math@gmail.com	

- **Do NOT call any number to ask if your submission has reached us.** **If you send your submission to the right email address with mentioned details, we will receive it. We will contact all participants who fail to submit, so please be patient.**

How to submit your answers

1. Write the following details **as per your registration** on the first page of your submission file/photographs -

Name	:	Your full name
School/College name	:	
Class/Year of Study	:	Class 8, Class 11, Undergrad 1st year etc.
Registered Email address	:	
Mode of S.T.E.M.S. Registration	:	Online / Through School / Other (mention details)

2. Write all your answers on sheets of paper, following all the solution guidelines. Write the page number in the top right corner of every sheet.

3. Scan your answers or take clear photographs of your response sheets. Compile them into a single PDF file or send all pictures in the right sequence.

   If you have a limited file size issue when sending your submission, make a new Google Drive folder with title '**Your Name - Mathematics B submission**'.

4. Send the submission file or Google Drive link from your registered email -

Submission email address	:	**stems.2020.math.b@gmail.com**
Subject of email	:	Mathematics Category B Submission - STEMS
Submission deadline	:	26th January 2020 - 18:00

Good luck, happy problem-solving!

Objective Questions

For **Problems 1-15**, each problem has **four** options, namely **(a), (b), (c), (d)**, of which ***only one*** is correct, **2 point** will be awarded for correctly answering a problem, **NO** negative marks shall be awarded for wrong answers/unattempted problems. The last seven problems of this section, **9-15**, are of "One value" type, answers are supposed to submitted as one value against the problem number in your submission.

Problem 1. *Find the number of integral solutions* (x, y) *to the following equation:*
$xy(x+y+1) = 2019^{2020} + 1$
a. 2019
b. 1
c. 2020
d. 0

Problem 2. *Find* $\lim_{N\to\infty} \frac{\sum_{n=0}^{N} \sin^4(5n)}{N}$.
a. $\frac{1}{16}$
b. $\frac{1}{2}$
c. $\frac{9}{16}$
d. $\frac{3}{8}$

Problem 3. *Let* $r_4(n)$ *be the number of ways you can write* n *as sum of four squares of non-negative integers. It's known (Lagrange's four squares theorem) that* $r_4(n) > 1$. *Find* $\lim_{n\to\infty} \frac{r_4(1) + \cdots + r_4(n)}{n^2}$.
a. $\frac{\pi^2}{32}$
b. $\frac{\pi^2}{2}$
c. $\frac{\pi^2}{8}$
d. $\frac{\pi^2}{243}$

Problem 4. $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ *such that* $f(0, x) = f(x, 0) = 1$ *for all* $x \in \mathbb{N} \cup \{0\}$. *Also for all* $i, j \in \mathbb{N}$ *we have* $f(i, j) = f(i-1, j-1) + f(i, j-1)$. *Find* $f(1009, 2019)$.
a. 2^{2018}
b. 2^{2020}
c. 2^{2019}
d. 2^{2021}

Problem 5. *Let* $*$ *be a binary operation on* $\mathbb{Z}^2$ *defined as follows:*

$(a,b)*(c,d)=(ac-2bd,bc+ad)$
Find the number of solutions to the equation: $(a,b)*(c,d)=(3,7)$
a.16
b. ∞
c. 32
d.0

Problem 6. *Let* $a,b\in\mathbb{N}$ *with* $gcd(a,b)=1$. *Find the number of ordered pairs* (a,b) *such that:*
$\frac{a}{b}+\frac{201b}{10201a}\in\mathbb{N}$
a. 1
b. 2
c. ∞
d. 0

Problem 7. *Find the number of solutions(in* $\mathbb{N}$*) to the following equation:*
$\phi(n^4+1)=8n$
c. 2
a. 1
d. 4
b. ∞

Problem 8. $X_1,X_2\cdots X_{13}$ *is regular* 13 *gon inscribed in a circle* Γ. *Let* $A_0=X_1,B_0=X_2,C_0=X_4$. *Given* $A_iB_iC_i$ *we construct* $A_{i+1}B_{i+1}C_{i+1}$ *in following way: Let* A_i *altitude in triangle* $A_iB_iC_i$ *meets* Γ *at* A_{i+1}. *Define* B_{i+1},C_{i+1} *in similar way. Find minimum* $i>0$ *such that* $A_i=A_0,B_i=B_0,C_i=C_0$.
a. 7
b. 6
c. 13
d. 12

Problem 9. *There are* m *trolls and* n *dwarves on the left side of a river. They have one boat to reach the opposite side of the river. They can use it as many times they want to reach to the opposite side, but a troll/dwarf can row the boat back from right side to left side only once. Also two trolls can't be on the boat at any time. Determine all pairs* (m,n) *with* $0\le m,n\le 2020$ *for which everyone can reach the opposite side.*

Problem 10. *Let* ABC *be a triangle inscribed in circle* Γ. *Let* $AB=2,AC=3,BC=4$ *and* D,E *be points on rays* AB *and* AC *such that* $AD=6,AE=4$. *Let* F *be the midpoint of* DE *and line* AF *meet* Γ *at* $G\neq A$. *Determine the ratio* $[GAB]:[GAC]$.

Problem 11. *Let* ABC *be a triangle circumcircle* Γ *and incircle* ω. *Let* Ω *be the circle tangent to* Γ *and the sides* AB, BC, *and let* $X=\Gamma\cap\Omega$. *Let* Y, Z *be distinct points on* Γ *such that* XY, YZ *are tangent to* ω. *Suppose radius of* ω *is* 1, *and radius of* Γ *is* R, *and* *angle $BAC = 60^\circ$. Determine YZ^2 in terms of R.*

Problem 12. *Let ABC be a triangle with $AB = 4, AC = 9$. Let the external bisector of angle A meet the circumcircle of triangle ABC again at $M \neq A$. A circle with center M and radius MB meets the internal bisector of angle A at points P and Q. Determine the length of PQ.*

Problem 13. *Let $\mathcal{H}$ be a rectangular hyperbola. Points A, B, C lie on $\mathcal{H}$. The incircle of triangle ABC touches sides AB, AC at points F and E, respectively. Suppose $\mathcal{H}$ intersects the segment EF at point P only. Let $AB = 13, BC = 14, CA = 15$. Determine the ratio $BP : PC$.*

Problem 14. *In triangle ABC with $\angle A = 60^\circ$, $AB = 4, BC = 8$, let ω denote the nine-point circle. Let AX, AY be tangents from A to ω. Determine XY^2.*

Problem 15. *Let ABC be a triangle inscribed in circle Γ with radius 1. Let M, N be the feet of internal bisectors of angles B and C. Line MN meets Γ again at P and Q, and lines BM and CN meet at I. Determine the radius of the circumcircle of triangle IPQ.*

Subjective Problems

This section contains **EIGHT** problems, of which, **ONLY SIX** of your best solutions shall be considered in your total score.

Problem 1. *Find all positive integers n for which we can select 2^{n-1} distinct subsets $S_1, \cdots, S_{2^{n-1}}$ of $\{1, 2, \cdots, n\}$ satisfying the following condition: for distinct $1 \leq i, j \leq 2^{n-1}, S_i \cap S_j \neq \phi \implies |i - j| \geq 4$.*

Problem 2. *Prove that every subgroup of $\mathbb{Q}$, which is a free-abelian group, is generated by a single element.*

Problem 3. *Let $S(a) := \{a^i + a^j | i, j \in \mathbb{N}\}$. Find all tuples (a, b, f) with $a, b \in \mathbb{N}$ and $f \in \mathbb{R}[x]$ such that $f(S(a)) \subset S(b)$*

Problem 4. *In triangle ABC with incenter I, the incircle ω touches sides $\overline{AC}$ and $\overline{AB}$ at points E and F, respectively. A circle passing through B and C touches ω at point K. The circumcircle of $\triangle KEC$ meets $\overline{BC}$ at $Q \neq C$. Prove that $\overline{FQ} \parallel \overline{BI}$.*

Problem 5. *Given a permutation $\sigma \in S_n$ of n objects, we construct a directed graph $G(\sigma)$ associated with it as follows. The vertices of this graph are all n-bit strings in $\{0, 1\}$, and there is an edge from s_1 to s_2 iff $\sigma(s_1) = s_2$. For example, if $\sigma \in S_5$ is the permutation for 5 elements which reverses the order of elements, then $\sigma(11010) = 01011$, so there is a directed edge from 11010 to 01011. (The permutations act on strings by σ maps the string $\sigma(a_1 a_2 \cdots a_n) = a_{\sigma^{-1}(1)} a_{\sigma^{-1}(2)} \cdots a_{\sigma^{-1}(n)}$). We say $\sigma_1 \sim \sigma_2$ for $\sigma_1, \sigma_2 \in S_n$ whenever $G(\sigma_1 \circ \sigma_2^{-1})$ has exactly 2^{n-1} self loops.*

- *Prove that given any $\sigma, \sigma' \in S_n$, there exist $\sigma_1, \cdots, \sigma_k \in S_n$, $k \in \mathbb{N}$, such that $\sigma \sim \sigma_1$, $\sigma_k \sim \sigma'$, and $\sigma_i \sim \sigma_{i+1}$ for each $i = 1, \cdots, k - 1$.*
- *Count the number of* ***ordered*** *pairs (σ_1, σ_2) where $\sigma_1 \sim \sigma_2$.*

Problem 6. *Let ABC be a triangle with incircle ω. Point D lies on the circumcircle of $\triangle ABC$. Denote by P and Q, the feet of perpendiculars from A, to the tangent lines from D to ω. Prove that line PQ passes through a fixed point independent of D.*

Problem 7. *Let $G = (V, E)$ be a bipartite undirected graph with partition $V = L(V) \cup R(V)$. For a subset $S \subseteq V$ let $N(S)$ denote the set of vertices adjacent to some vertex in S. Suppose for all distinct $x, y \in V$, there exists a $z \in V$ such that $z \in N(x), z \notin N(y)$. Define the score of a subset $S \subseteq V$ to be $s(S) = 2^{|S| - |N(S)|}$.*

Let T_0 be the sum of scores of all subsets of $L(V)$ with even size, i.e. $T_0 = \sum_{\substack{S \subseteq L(V) \\ |S| \text{ even}}} s(S)$. Let

T_1 be the sum of scores of all subsets of $L(V)$ with odd size, i.e. $T_1 = \sum_{\substack{S \subseteq L(V) \\ |S| \text{ odd}}} s(S)$. *Show that* $|T_0 - T_1| \leq 1$.

Problem 8. *Let R be an integral domain, $K(R)(\neq R)$ be the fraction field of R. Prove that any non zero $K(R)$-vector space* ***cannot*** *be a free R-module.*