TESSELLATE PRESENTS

STEMS

Scholastic Test of Excellence in Mathematical Sciences

Computer Science Category B

Exam Date	**:**	**25th January 2020**
Exam Timing	**:**	**12noon - 3pm**

Rules and Regulations

Marking Scheme

1. The question paper is divided in two parts -
   Objective (8 questions) + Subjective (5 questions).
2. Each objective question is worth **2 points**.
3. Each subjective problem is worth **10 points**.
4. There is no negative marking in any section.
5. **The subjective part will be graded only if you score above a certain cut-off (to be decided later) in the objective section of the paper**.
6. **For the final score, your total score (subjective + objective) will be taken into consideration.**

Solution guidelines

1. You are **NOT** required to show your work for the objective part of the paper.
   Only tick your option choices in a tabular format drawn on a blank sheet.
   A sample is shown below.

Q. No.	(a)	(b)	(c)	(d)
1		✓		
2			✓	
3		✓		
4	✓			
5				✓

Fig. - Sample objective answer submission

2. Provide a complete solution/proof for subjective questions. The solutions must be correct, original, detailed and clearly understandable for full credit.
   Partial credit might be awarded to incomplete proofs, based on the progress made towards solving the problem.
3. Draw clear, well-labeled diagrams wherever necessary.

Miscellaneous

1. Answers should be your own and should reflect your independent thinking process.
2. Do **NOT** post the questions on any forums or discussion groups. It will result in immediate disqualification of involved candidates when caught.
3. Answers should be written clearly, in a legible way. Formal proofs are required wherever asked for. Unclear reasoning might not be awarded points, draw clear diagrams wherever necessary.
4. Sharing/discussion aimed towards solving or distribution of problems appearing in the contest while the contest is live in any kind of online platform/forum shall be considered as a failure in complying with the regulations.
5. Any form of plagiarism or failure to comply with aforementioned regulations may lead to disqualification.

Contact details - ONLY for subject related queries

- Please do not call these people for technical problems or submission inquiries.
  Only if you find an ambiguity in a question and need clarification, use these contacts.
- As our phone numbers will be busy, **we prefer WhatsApp & email queries** instead. Only call us if absolutely necessary.
- Try to solve all your submission related doubts from information in the next page ONLY. We have included all details in the next section.

Rajat De - 9716506316
Query email - stems.2020.enq.cs@gmail.com

- **Do NOT call any number to ask if your submission has reached us.**
  If you send your submission to the right email address with mentioned details, we will receive it. We will contact all participants who fail to submit, so please be patient.

How to submit your answers

1. Write the following details **as per your registration** on the first page of your submission file/photographs -

Name	:	Your full name
School/College name	:	
Class/Year of Study	:	Class 8, Class 11, Undergrad 1st year etc.
Registered Email address	:	
Mode of S.T.E.M.S. Registration	:	Online / Through School / Other (mention details)

2. Write all your answers on sheets of paper, following all the solution guidelines. Write the page number in the top right corner of every sheet.

3. Scan your answers or take clear photographs of your response sheets. Compile them into a single PDF file or send all pictures in the right sequence.

   If you have a limited file size issue when sending your submission, make a new Google Drive folder with title '**Your Name - Computer Science B submission**'.

4. Send the submission file or Google Drive link from your registered email -

Submission email address	:	**stems.2020.cs.b@gmail.com**
Subject of email	:	Computer Science Category B Submission - STEMS
Submission deadline	:	25th January 2020 - 15:00

Good luck, happy problem-solving!

Objective Questions

For **Problems 1-8**, each problem has **four** options, namely **(a), (b), (c), (d)**, of which ***only one*** is correct, **2 point** will be awarded for correctly answering a problem, **NO** negative marks shall be awarded for wrong answers/unattempted problems.

Problem 1. *Given a graph G, let max-clique(G) denote the size of the largest clique which is a subgraph of G, $\chi(G)$ denote the chromatic number of G, max-ind(G) denote the size of the maximal independent set of G, and tree-width(G) denote the tree-width of the graph (refer to `https://en.wikipedia.org/wiki/Treewidth`). Which of the following may be false for a graph G?*
a. max-clique$(G) \leq \chi(G)$
b. tree-width$(G) \leq \chi(G)$
c. ceil$(|G|/$max-ind$(G)) \leq \chi(G)$
d. $\chi(G) \leq \max(degree(G)) + 1$

Problem 2. *For which of the following problems would having a polynomial time solution imply a polynomial time solution for 3-SAT (assuming $P \neq NP$)*
a. Counting the number of perfect matchings of a bipartite graph
b. Computing the permanent of an integer matrix modulo 2
c. Factoring integers
d. Checking if 2 graphs are isomorphic

Problem 3. *Consider the following game played between 2 players:*
Given an empty $1 \times n$ board, the players take turns alternatively placing a coin in one of the cells. No 2 adjacent cells can contain a coin while doing so. The player who cannot make a move loses.
For which value of n, can the second player force a win?
a. 65
b. 52
c. 58
d. 71

For the next two problems, let $\Sigma = \{a, b, c\}$. Let $|w|_a$ refer to the number of $a's$ in the word w. Let $L_1, L_2, L_3, L_4, L_5 \subseteq \Sigma^*$ be the following languages:
$L_1 = \{u.c^+.v \mid u, v \in \Sigma^*, |u|_a = |v|_b\}$
$L_2 = \{a^n b^n c^n \mid n \in \mathbb{N}\}$
$L_3 = \{ww \mid w \in \Sigma^*\}$
$L_4 = \{u.v \mid u, v \in \Sigma^*, |u|_a = |v|_a\}$
$L_5 = \{u.v \mid u, v \in \Sigma^*, |u|_a = |v|_b\}$

Problem 4. *Which of the following is correct:*
a. L_1, L_2, L_3 *are regular languages.*
b. L_2 *and* L_4 *are regular languages.*
c. L_4 *and* L_5 *are regular languages.*
d. L_1 *and* L_5 *are regular languages.*

Problem 5. *Which of the following is correct:*
a. L_1, L_2 *and* L_3 *are not context-free languages.*
b. L_1, L_2, L_3 *and* L_5 *are context-free languages.*
c. L_1, L_3^c, L_4 *and* L_5 *are context-free languages.*
d. L_1^c, L_3, L_4 *and* L_5 *are context-free languages.*

Problem 6. *Let* L, L_1, L_2 *be languages over* $\Sigma = \{a, b\}$. *Which of the following statements is true?*
a. $\{a\}^{-1}(\{a\}.L) = L$
b. $\{a\}(\{a\}^{-1}L) = L$
c. $L_1^{-1}(L_1.L_2) = L_2$
d. $L_1(L_1^{-1}L_2) = L_2$

Problem 7. *Let* $f(n)$ *be defined as* $f(n) = \frac{n}{2}$ *if* n *is even, and* $f(n) = 3n + 1$ *if* n *is odd. How many iterations of* f *do we need to apply to* 55 *to reach* 1*?*
a. 55
b. 69
c. 112
d. 182

Problem 8. *Consider a randomly generated tournament graph of n vertices (for each unordered pair of vertices (x, y) with $x \neq y$, a directed edge is chosen either from x to y or from y to x depending on an unbiased coin toss). What is the expected number of Hamiltonian paths in this graph?*

a. $\frac{n!}{\pi^{n-1}}$

b. $\frac{n!}{2^{n-1}}$

c. $\frac{n!}{e^{n-1}}$

d. $\frac{n!}{\varphi^{n-1}}$, *where* $\varphi = \frac{1+\sqrt{5}}{2}$

Subjective Problems

Problem 1. *Consider a partially ordered set and define a graph on the poset by joining x with y only if they are comparable in the poset. Prove that in this graph, $\chi(G) =$ max-clique(G).*

Problem 2. *$G = (V, E)$ is a graph. Consider an element $x \in \mathbb{F}_2^{|V|}$. For a vertex v, let $S_v \subseteq V$ denote $S_v := \{u | u = v$ or $(u, v) \in E)\}$. Let $I_{S_v} \in \mathbb{F}^{|V|}$ denote its characteristic vector. Suppose for any v you are allowed to take the XOR of x with I_{S_v}. Show that if you start with $x = 0^{|V|}$, there is a sequence of such operations that can be done to reach $x = 1^{|V|}$.*

Problem 3. *Let $G = (V, E)$ be a graph. Suppose there is an injective function $f : V \to \mathbb{R}$. We define a weight function $g : E \to \mathbb{R}$ as $g(u, v) = (f(u) - f(v))^2$. Suppose now we keep removing edges one by one in non-decreasing order of their weights till the graph becomes disconnected. Suppose A and B are the two partitions of G thus obtained. Prove that either* $\max(A) < \min(B)$ *or* $\max(B) < \min(A)$.

Problem 4. *Consider the following definitions.*

Definition 1. *A matrix $M \in \mathbb{R}^{k\times n}$ is said to have the* ***restricted isometry property*** *with parameters (r, δ_r) if*

$$(1 - \delta_r) \cdot ||x||^2 \leq ||Mx||^2 \leq (1 + \delta_r) \cdot ||x||^2$$

for all $x \in \mathbb{R}^n$ which satisfy $|\{i \mid x_i \neq 0\}| \leq r$. □

Definition 2. *Let $M \in \mathbb{R}^{k\times n}$ be such that $||M^{(i)}|| = 1$, for each column $M^{(i)}$ of M, for all $1 \leq i \leq n$. Define* ***coherence*** *of M as :*

$$\mu(M) = \max_{i\neq j} |\langle M^{(i)}, M^{(j)}\rangle|$$

□

Suppose $M \in \mathbb{R}^{k\times n}$ be such that $||M^{(i)}|| = 1$ for each column $M^{(i)}$ of M. Then show that for each r, M has the restricted isometry property with parameters $(r, (r - 1)\mu(M))$.

Problem 5. *Construct a partially ordered set that has no infinite anti-chain but cannot be covered by finitely many chains.*